ON T-CONVEXITY, NON-PFAFFIANNESS AND DIFFERENTIAL FIELDS

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By CHRISTOPH KESTING, M.Sc.

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AUTHOR:	Christoph Kesting M.Sc. Mathematics, University of Münster, Münster, Germany
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Lay Abstract

In model theory, we study mathematical structures in relation to what we allow to be expressible in their languages. This follows the motto ”If you talk about less, you can show more meaningful results.” In this thesis, we study how fields like the real and complex numbers change when we introduce generalizations of distance or differentiation to the language. We show that in many cases, the resulting structures remain well behaved, but in special cases, this can fail. Further, we show that the Klein-j-function, a prominent object of study in number theory, is not expressible in the structure of the real field with the exponential function and a pfaffian chain, which is a list of solutions to successive order one differential equations. This gives an indication that we need a different approach to address certain number theoretic problems with the tools of model theory.

Abstract

We study the model theoretic expansions of certain fields by valuations, derivations, and pfaffian chains. In particular, we show that o-minimal expansions of real fields, equipped with a T-convex valuation and a monomial group of representatives, are well behaved if and only if an exponential function is not definable. Similarly, we show that differential expansions of éz-fields by a generic derivation are the same as taking the differentially large expansion. Further, we show that these expansions preserve certain model theoretic tameness properties, including NTP_2. Further, we study the field extensions of the real exponential field by a pfaffian chain (a set of solutions to a triangular system of order one differential equations) and show that a real restriction of the Klein j-function is not $\mathbb{R}_{\exp}$-definable, indicating that these extensions are insufficient for certain applications in number theory.

to Renate, Klaus, Martin, Bea, Tessa and Till

who have always supported me and given me strength

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Chapter 1

Introduction

In the study of model theory, it is a common theme to ask the two questions "How far can I expand a given structure, without it losing tameness?" and "Is it possible to answer a specific question in a given structure?" Throughout this thesis, we aim to answer both of these questions in three connected but distinct areas of model theory.

The first area we tackle is in the study of T-convex valued fields. These are expansions of o-minimal fields with a theory T equipped with a compatible convex valuation ring. They were first studied by van den Dries and Lewenberg [vdDL95] and later expanded by van den Dries in [vdD97]. These valued fields are prototypical models of asymptotic fields, as the valuation can be arranged to mirror the Landau notation of Big O and little o for asymptotic growth. For o-minimal structures, Miller's dichotomy says that either an exponential function is definable, or every function is eventually bounded by a power function. Following this spirit, van den Dries and Lewenberg established multiple structural results mirroring classical results in the study of valued fields for power bounded valued fields, like relative quantifier elimination. However, in the exponential case, similarly strong results have not been established.

In Chapter 4, we expand on this in joint work with Elliot Kaplan, by adding a monomial group to the T-convex expansions of o-minimal fields. A monomial group $\mathfrak{M}$ is a subgroup of the positive multiplicative group $\mathcal{R}$ of the valued field, on which the restriction of the valuation is a group isomorphism. Adding a monomial group is oftentimes necessary or aids in the process of establishing a quantifier elimination result. In the case that T is power bounded, we show that the T-convex valued field with a compatible monomial group enjoys almost the same tameness properties as our original field $\mathcal{R}$, see Theorems 4.2.5 for quantifier elimination, Theorem 4.2.11 for a characterization of one-dimensional definable sets and Theorems 4.2.13 and 4.2.12 for NIP and distality. In contrast, if T defines an exponential function, then $\mathbb{N}$ is externally definable in any model of T with a monomial group that is compatible with the exponential, see Corollary 4.2.16. This, of course, indicates that the expansion in the exponential case is not tame in any meaningful way. This is strong evidence that the lack of tameness results for exponential T-convex valued fields without a monomial group might be due to a lack of tameness.

The second area of study, cotinues with the theme of o-minimal structures and their expansions. Expansions of o-minimal structures by pfaffian chains have been an intense subject of study. Based on the ideas of Khovanskii [Kho91], model theorists have been studying expansions of o-minimal structures by pfaffian chains and, more generally, Rolle leaves. Pfaffian chains of functions are defined as solutions of order 1 differential equations, see definition 5.2.1 and [Kho91, Section 2.3]. In contrast, the general theory of Khovanskii is captured by the study of nested Rolle leaves [LS10], which are geometrically more involved and in particular, capture the notion of pfaffian chains. Since the inception of these two notions, it has been an open question whether they are equivalent (and in what sense).

In Chapter 5, we approach this question using the modular j-function. It has been commonly known that the inverse of the j-function is pfaffian [Arm20]. Freitag showed in [Fre21] that the j-function does not belong to a pfaffian chain over the complex numbers. In contrast, systems of nested Rolle leaves are known to be closed under inverse functions, showing that the graph of j belongs to a nested Rolle leaf, indicating a gap between the notions of nested Rolle leaves and pfaffian chains. Following this impetus, we show that the real-valued restriction $j(ix)$ to the imaginary axis of the j function does not belong to a pfaffian chain over the real exponential field $\mathbb{R}_{\exp}$, see Theorem 5.2.16. This shows that these notions of pfaffian chain and nested Rolle leaves do not agree even in this first-order definable setting over $\mathbb{R}_{\exp}$.

Another motivation for the study of the j function relates to the study of one of the most significant applications of o-minimality to unlikely intersection problems, largely motivated by questions in diophantine geometry. These involve applying the Pila-Wilkie point counting theorem [PW06], in various o-minimal structures, in order to estimate the number of solutions of various systems of equations. One particular application is Pilas' proof of the Andre-Oort conjecture [Pil11], involving the modular j-function. The proof used the o-minimal structure $\mathbb{R}_{an,\exp}$, which contains both exp and a restriction of the modular j-function used to classify elliptic curves. Wilkie conjectured, and Binyamini, Novikov and Zak [BNZ24] later proved, that certain o-minimal expansions, like $\mathbb{R}_{\exp}$ or expansions of the reals by restricted $\mathbb{R}$-pfaffian chains, admit point counting results with improved bounds. In particular, the non-pfaffian result of j for $\mathbb{R}_{\exp}$ Theorem 5.2.16 indicates that the method for $\mathbb{R}_{\exp}$ does not translate for the unrestricted real j-function.

For the third area, we turn our attention towards the extension of fields K by derivations δ. In particular, we ask the question of when these resulting differential fields are tame. Heuristically, the model theory of a tame differential field (K, δ) should be determined by the model theory of K, as long as δ is suitably generic. To verify this, in joint work with Elliot Kaplan, we compare the frameworks of generic derivations on algebraically bounded fields introduced in [FT24] and differentially large fields introduced in [LST24], which serve as a differential analogue to large fields introduced by Pop [Pop14]. In the process, we show that these two notions coincide on éz-fields introduced in [WY23]. Beyond that, we are concerned with the extent to which any tameness properties translate from the underlying field K to the differential field extension (K, δ). Many common model theoretic properties have been shown to hold after adding a generic derivation, including quantifier elimination, stability, NIP and simplicity (see Fact 6.4.1 for more). We add to this list by showing that an algebraically bounded NTP_2 structure K remains NTP_2 after expanding by a generic derivation, and similarly for structures avoiding the antichain tree property introduced by [AKL23], using the NTP_2 proof as a blueprint. We expect the proof technique to generalize to further properties based on an inconsistency pattern.

1.1 Overview of results

In Chapters 2 and 3, we give an overview of relevant results in the realms of stability theory, neo-stability and o-minimality relevant to our work and introduce the theory of differentially closed fields.

In Chapter 4, we present joint work with Elliot Kaplan in [KK24] on T-convex expansions of o-minimal fields with a monomial group, where T is the theory of $\mathcal{R}$. We show, following the spirit of Miller's Dichotomy 3.2.3, a dichotomy for these expansions. If T is power bounded and as such does not define an exponential function, then the T-convex valued field with a compatible monomial group enjoys almost the same tameness properties as our original field $\mathcal{R}$. If, however, T does define an exponential function, then $\mathbb{N}$ is externally definable in any model of T with a monomial group that is compatible with the exponential. This, of course, indicates that the expansion in the exponential case is not tame in any meaningful way.

In Chapter 5, we study the definability of the j-function over $\overline{\mathbb{R}}$- and $\mathbb{R}_{\exp}$-pfaffian chains, building on Freitags argument for $\mathbb{C}$-pfaffian chains in [Fre21]. We show that a suitable real-imaged slice of j, called $\mathfrak{j}$ is not definable over either of these types of pfaffian chains. As the proof relies on the strong minimality of the differential equation of j, we give an account of Scanlon's and Freitag's proof of this fact from [FS18].

In Chapter 6, we again present joint work with Elliot Kaplan [KK26], on a comparison between the frameworks of generic derivations on algebraically bounded fields and differentially large fields. In the process, we show that these two notions coincide on éz-fields. Further, we show that an algebraically bounded NTP_2 structure remains NTP_2 after expanding by a generic derivation.

Chapter 2

Stability and Neo-Stability

Before we truly begin, we first need to introduce some concepts from the realm of model theory, in particular, stability theory and the further development into neo-stability theory. We will first give a brief description of stability, forking and the U-rank, and introduce the theory of differentially closed fields in characteristic 0 as our main example.

For this section, we will be working in a many-sorted language $\mathcal{L}$, a theory T, and an ambient monster model $\mathcal{U} \models T$.

2.1 Stability

For a full introduction to stability theory, we recommend [Pil96] for more details. Instead, we aim to cover the definitions that might go beyond a first course in model theory.

Definition 2.1.1. T is stable if there is no infinite set $\{a_i : i \in \mathbb{N}\} \subseteq \mathcal{U}^n$ and a formula

$\phi(x, y)$ such that

$$\mathcal{U} \models \phi(a_i, a_j) \Leftrightarrow i < j.$$

Notable examples of stable theories are algebraically closed fields (ACF), see [MMP05], and separably closed fields (SCF), see [Woo79].

In particular, we will be focusing on the theory of differentially closed fields of characteristic 0.

Example 2.1.2. A derivation on a field K is a K-additive map $\delta : K \to K$ that satisfies Leibniz's law, i.e. for $a, b \in K$ we have that $\delta(ab) = a\delta(b) + \delta(a)b$. The pair (K, δ) is called a differential field, and the set $C = \{a \in K | \delta(a) = 0\}$ is called the field of constants. We usually consider differential fields in the language $\mathcal{L}_\delta = \mathcal{L}_{ring} \cup \{\delta\}$. The theories of differential fields in characteristic 0 have a model companion called DCF_0, the theory of differentially closed fields, see [MMP05].

Within stable structures, there is a canonical notion of saying "A is independent of B over C", called forking, which is defined as follows:

Definition 2.1.3. We say a formula $\phi(x, a)$ divides over a set C if there exists a C-indiscernible sequence $a_1, a_2, \ldots$ of realizations of $\mathrm{tp}(a/C)$ such that $\{\phi(x, a_i) : i \in \mathbb{N}\}$ is inconsistent. Similarly, a formula $\phi(x, a)$ forks over a set C if it implies a finite disjunction $\bigvee_{j=1}^{n} \psi_j(x, b_j)$ of formulas with parameters b_j from $\mathcal{U}$, with each $\psi_j(x, b_j)$ dividing over C. A partial type $p(x)$ forks (or divides) over C if some formula in it forks (or divides) over C.

We then define

$$A \mathop{\downarrow}_C^f B$$

to mean that $\mathrm{tp}(a/BC)$ does not fork over C for every finite tuple a from A.

Using forking, we can define the following rank on types:

Definition 2.1.4. Let p be a n-type over some set A. We define the U-rank as follows:

- $U(p) \geq 0$
- If λ is a limit ordinal then $U(p) \geq \lambda$, if $U(p) \geq \alpha$ for all $\alpha < \lambda$.
- If $\alpha = \beta + 1$, we say $U(p) \geq \alpha$, if there is a forking extension q of p that satisfies $U(q) \geq \beta$.

We say $U(p) = \alpha$ if and only if $U(p) \geq \alpha$ but not $U(p) \geq \alpha + 1$. If $U(p) \geq \alpha$ for all ordinals α, then we write $U(p) = \infty$.

In DCF_0, the situation is as follows:

Fact 2.1.5 ([MMP96]). *Given a set $A \subset \mathcal{U}$, let $\mathrm{acl}_\delta(A)$ be the algebraic closure of the differential field generated by A, i.e. the model theoretic algebraic closure in $\mathcal{L}_\delta$. If $\overline{a}$ is a tuple, let $\mathrm{trdeg}(\overline{a}/A)$ be the transcendence degree of $\overline{a}$ over the (usual) algebraic closure $\mathrm{acl}(A)$. Here, the following holds:*

$$A \downarrow^f_C B \Leftrightarrow \forall \overline{a} \in \mathrm{acl}_\delta(AC), \mathrm{trdeg}(\overline{a}/\mathrm{acl}_\delta(C)) = \mathrm{trdeg}(\overline{a}/\mathrm{acl}_\delta(BC))$$

This notion of forking coincides with the standard notion of forking in DCF_0, see [MMP96].

Furthermore, we need to introduce:

Definition 2.1.6. In a stable theory with elimination of imaginaries, such as DCF_0, a canonical base $cb(p)$ of a type $p(x) = tp(a/B)$ is a set C, such that for all automorphisms σ of the monster model $\mathcal{U}$ we have:

$$\sigma \text{ fixes } p(x) \Leftrightarrow \sigma \text{ fixes } C \text{ pointwise}$$

Definition 2.1.7. A set A defined by a formula φ is strongly minimal if every definable subset of A is either finite or cofinite. Further, a theory T is strongly minimal if the formula $x = x$ is strongly minimal.

Equivalently, this is the same as saying that the U-rank of the partial type $\{\varphi\}$ is 1.

Definition 2.1.8. A theory is ω-stable if the U-rank for every partial type is finite.

Fact 2.1.9 ([TZ, Corollary 6.4.7])**.** *DCF_0 is ω-stable, so in particular DCF_0 is superstable and stable.*

Definition 2.1.10. In a stable theory with elimination of imaginaries, a type $tp(a/A)$ is called stationary if it has only one unique non-forking extension for each $B \supseteq A$. This is in particular the case if $A = acl_{\mathcal{L}}(A)$. A Morley sequence in a stationary type p is any indiscernible sequence $(a_i)_{i<\kappa}$, κ an ordinal, such that each a_i is a realisation of the unique non-forking extension over $\{a_j : j < i\}$.

The following fact is often referred to as the Shelah reflection principle.

Fact 2.1.11 ([Pil96, Lemma 2.28/3.19])**.** *In a stable theory T, let $p(x)$ be a stationary type and $(a_i : i < \omega)$ be a Morely sequence in $p(x)$. Then the canonical base $Cb(p) \subseteq dcl(a_i : i < \omega)$. If T is superstable then there is $n < \omega$ such that $Cb(p) \subseteq dcl(a_i : i < n)$.*

2.2 Neo Stability

Outside the realms of stability, we occasionally can find sets that are behaving in a stable way. Let $\mathcal{R} \models T$.

Definition 2.2.1. A definable set D in $\mathcal{R}$ is stably embedded if every $\mathcal{R}$-definable subset of D^n, for any n, is D-definable. In the case that D is a sort, we say that D is purely stably embedded if the induced structure on D is already given by the restriction of $\mathcal{L}$ to D.

Note that in stable theories, every definable set is stably embedded, as a consequence of the definability of types in stable theories [Pil96, Lemma 2.2].

Definition 2.2.2. For a theory T, two sorts P and Q, having tuples of variables x and y respectively, are *orthogonal* if any formula $\phi(x,y)$ is equivalent to a boolean combination of multiple formulas of the form $\psi(x)$ and $\theta(y)$.

Definition 2.2.3. A formula $\phi(x,y)$ has IP (the independence property) if there is an infinite Set A of $|x|$-tuples that is shattered by $\phi(x,y)$, i.e. we can find a family $(b_I : I \subseteq A)$ of $|y|$-tuples such that

$$\mathcal{U} \models \phi(a, b_I) \Leftrightarrow a \in I, \text{ for all a in A.}$$

A formula is NIP if it doesn't have IP. A theory is NIP if every formula is NIP.

Distality was introduced by Pierre Simon in [Sim13], which captures an order-like behaviour within NIP theories.

Definition 2.2.4. A NIP theory T is called distal if for every indiscernible sequence $(a_i)_{i \in I}$ and any parameter set A such that

- I can be split into $I = I_1 + c + I_2$ (i.e. $I_1 < c < I_2$ and $I = I_1 \cup \{c\} \cup I_2$), where I_1 and I_2 are infinite and without endpoints and
- if $(a_i)_{i \in I_1 + I_2}$ is A-indiscernible, then the entire sequence $(a_i)_{i \in I}$ is A-indiscernible as well.

Noteable examples of such theories are all o-minimal theories, Presburger arithmetic $(\mathbb{Z}, 0, 1, +, <)$, the theory of the p-adics $\mathbb{Q}_p$ as a valued field [DGL11] and the transseries $\mathbb{T}$ [AvdHVdD17].

Definition 2.2.5. We say a theory T is strongly dependent if there are no formulas $\phi_i(x, y)$ and parameters $a_{i,j}$ with $i, j < \omega$ such that for every $f : \omega \to \omega$ the type $\{\phi_i(x, a_{i,f(i)} : i < \omega\}$ is consistent, but $\{\phi_i(x, a_{i,j} : j < \omega\}$ is k-inconsistent for some $k \in \mathbb{N}$, see [Pil13b] for more details.

2.2.1 NTP$_2$

Definition 2.2.6. A formula $\varphi(x, y)$, in tuples x and y, has the *tree property of the second kind* (TP$_2$) if there is an array of tuples $(a_{i,j})_{i,j<\omega}$ of length $|y|$ and a $k \in \mathbb{N}$ such that:

1. The partial type $\{\varphi(x, a_{i,j})\}_{j<\omega}$ is k-inconsistent for all i.
2. The partial type $\{\varphi(x, a_{i,f(i)} : i < \omega\}$ is consistent for all functions $f : \omega \to \omega$.

A theory T has TP$_2$ if some formula has TP$_2$, and T has NTP$_2$ if it does not have TP$_2$.

First we have the following consequence of [Che14, Corollary 2.9 and Lemma 3.2]:

Fact 2.2.7. *If T is TP_2, then there is a formula $\varphi(x,y)$ with x being a single variable witnessing it.*

Fact 2.2.8 ([CH14, Definition 3.1 and Lemma 3.9])**.** *We say that $(c_{ij})_{i,j\in\kappa}$ for a cardinal κ is a strongly indiscernible array if $\overline{c}_i = (c_{ij})_{j\in\kappa}$ is indiscernible over $(\overline{c}_l)_{l\neq i}$ for all i and $(\overline{c}_i)_{i\in\kappa}$ is an indiscernible sequence (of sequences). In any theory, $\varphi(x,y)$ has* TP_2 *if and only if there is such a strongly indiscernible array $(a_{ij})_{i,j\in\omega}$ witnessing it and $c \vDash \{\varphi(x, a_{i0})\}_{i\in\omega}$ such that the sequence of rows $(\overline{a}_i)_{i\in\omega}$ is indiscernible over c.*

2.2.2 NATP

Another inconsistency pattern like TP_2 is the antichain tree property. For this, we first introduce some notation.

On branches of finite length on the infinite binary tree $\eta,\nu \in 2^{<\omega}$ we define

- $\eta \triangleleft \nu$ for η being a strict truncation of ν and
- $\eta^\frown\nu$ for the concatenation of η and ν. Furthermore given $A \subseteq 2^{<\omega}$ we define $\eta^\frown A = \{\eta^\frown\nu : \nu \in A\}$.

Following [AKL23, Definition 2.4], we say that a tree $(a_\eta)_{\eta\in\kappa^{<\lambda}}$, for cardinals κ and λ is strongly indiscernible if the type of tuples $\mathrm{tp}(\overline{a}_{\overline{\eta}}) = \mathrm{tp}(\overline{a}_{\overline{\nu}})$ for all $\mathrm{qftp}_0(\overline{\eta}) = \mathrm{qftp}_0(\overline{\nu})$ in the language $\mathcal{L}_0 := \{\triangleleft, <_{lex}, \wedge\}$ of ordered meet trees. Further, $(b_\eta)_{\eta\in\kappa^{<\lambda}}$ is strongly locally based on $(a_\eta)_{\eta\in\kappa^{<\lambda}}$ if for all tuples $\overline{\eta}$ and a finite set of $\mathcal{L}$-formulas Δ, there is $\overline{\nu}$ such that $\mathrm{qftp}_0(\overline{\eta}) = \mathrm{qftp}_0(\overline{\nu})$ and $\overline{b}_{\overline{\eta}} \equiv_\Delta \overline{a}_{\overline{\nu}}$.

Fact 2.2.9 ([AKL23, Fact 2.5])**.** *Let a tree-indexed set $(a_\eta)_{\eta\in\omega^{<\omega}}$ be given. Then there is a strongly indiscernible $(b_\eta)_{\eta\in\omega^{<\omega}}$ which is strongly locally based on $(a_\eta)_{\eta\in\omega^{<\omega}}$.*

This statement is called the *modelling property of strong indiscernibility*, and is one of the key ingredients for the analogous result for the property below.

Definition 2.2.10. A formula $\varphi(x, y)$ in tuples x and y has the *k-antichain tree property* (k-ATP or just ATP) for some $k \in \mathbb{N}$ if there exists a tree indexed set of tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ of length $|y|$ such that:

1. The partial type $\{\varphi(x, a_\eta)\}_{\eta \in C}$ is k-inconsistent whenever C is a $\triangleleft$-chain.

2. The partial type $\{\varphi(x, a_\eta : \eta \in A\}$ is consistent for any antichain $A \subseteq 2^{<\omega}$.

A theory T has ATP if there is a formula having ATP and T is NATP is T is not ATP.

First note that the distinction between k-ATP and ATP is unnecessary, similar to the TP_2-case.

Fact 2.2.11 ([AKL23, Lemma 3.20])**.** *A complete theory T has k-ATP for some $k \geq 2$ if and only if T has ATP.*

In the same way, there is also a 1-variable theorem:

Fact 2.2.12 ([AKL23, Theorem 3.17])**.** *If there exists a witness of ATP, then there exists a witness of ATP in a single free variable.*

Note that ATP already implies TP_2 and SOP_1, see [AK24, Propositions 2.4 and 2,6], so most commonly studied structures in model theory have NATP.

Chapter 3

O-minimal basics

In this chapter, we will give an overview of the foundational results in the study of o-minimal structures. We first give a Definition, Examples, and introduce the rank of an o-minimal structure. Then in Section 3.1, we introduce the concept of o-minimal cell decomposition, and its analytic counterpart. Lastly, in Section 3.2, we introduce Miller's dichotomy between being a structure able to define an exponential function or being power bounded.

Let $\mathcal{L} \supseteq \{<\}$ be a language containing a binary relation $<$ and let $\mathcal{R}$ be an $\mathcal{L}$-structure expanding the theory of a dense linear order without endpoints on the base set R. If all of the $\mathcal{L}$-definable subsets of $\mathcal{R}$ are finite unions of intervals and points, then $\mathcal{R}$ is called an *o-minimal structure.* Let T be a theory extending the theory of dense linear orders without endpoints. T is called an *o-minimal theory*, if every model of T is o-minimal. Throughout this thesis, we are working with o-minimal theories extending the theory of real closed ordered fields and expanding the language of ordered rings $\mathcal{L}_{o-ring} = \{0, 1, +, -, \cdot, <\}$, and o-minimal structures $\mathcal{R}$ expanding on underlying fields R.

Remark 3.0.1. *We note that an o-minimal theories expanding a dense ordered abelian group with one non-zero constant symbol has definable choice, meaning that for every formula $\phi(x,y)$ there is a function $f(y)$ such that whenever $\mathcal{R} \models T$, $b \in \mathcal{R}$, and $\phi(R,b)$ is non-empty, then $f(b) \in \phi(R,b)$, further, $f(b)$ depends only on $\phi(R,b)$, hence if $\phi(R,b) = \phi(R,b')$, then $f(b) = f(b')$.*

Oftentimes, we will use this fact to pass to the Skolemisation of T, by extending and closing the language $\mathcal{L}$ by all choice functions f as above. This has the advantage that T in the extended language has quantifier elimination without changing the definable sets, as all added functions were definable in the first place.

Example 3.0.2. Besides this general setting above, we will refer to the following o-minimal structures throughout this thesis:

1. $\overline{\mathbb{R}}$ is the o-minimal structure of real numbers $\mathbb{R}$ in the language $\mathcal{L}_{o-ring}$, see [TM51].

2. $\mathbb{R}_{\exp}$ is the expansion of $\overline{\mathbb{R}}$ by the exponential function $\exp : \mathbb{R} \to \mathbb{R}$, see [Wil96].

3. $\mathbb{R}_{an}$ is the expansion of $\overline{\mathbb{R}}$ by restricted analytic functions, meaning that $\mathcal{L}_{o-ring}$ is expanded by a function symbol f for each $n \in \mathbb{N}$ and each analytic function $f : [0,1]^n \to \mathbb{R}$, see [vdDMM94].

4. $\mathbb{R}_{an,\exp}$ is the expansion of $\mathbb{R}_{an}$ by the exponential function, also see [DvdD88] .

In particular, we note:

Fact 3.0.3 ([Wil96])**.** $\mathbb{R}_{\exp}$ *is model complete, i.e., every first-order definable set is equivalent to a projection of a quantifier-free definable set.*

Furthermore, o-minimal structures allow for a dimension theory.

Definition 3.0.4. Let $\mathcal{S}$ be an extension of $\mathcal{R}$ of o-minimal structures in the language $\mathcal{L}$. Then the rank $\mathrm{rk}_{\mathcal{L}}(\mathcal{S} \mid \mathcal{R})$ of $\mathcal{S}$ over $\mathcal{R}$ is defined as the minimal cardinality of a set of generators of $\mathcal{S}$ over $\mathcal{R}$. We denote for a tuple c we denote by $\mathcal{S} = \mathcal{R}\langle c \rangle$, that $\mathcal{S}$ is the o-minimal structure generated by taking the definable closure of $c \cup \mathcal{R}$.

3.1 Cell decomposition

Definition 3.1.1. A cell C is a definable subset of R^k, which is constructed inductively as follows:

1. Singletons and open intervals are cells.
2. Given a cell $C \subseteq R^n$, if $f, g : C \to R$ are definable continuous functions such that $f < g$ on C, then

$$\begin{aligned} (f, g) &:= \{(x, r) \in C \times R : f(x) < r < g(x)\}, \\ (-\infty, f) &:= \{(x, r) \in C \times R : r < f(x)\}, \\ (f, +\infty) &:= \{(x, r) \in C \times R : f(x) < r\} \end{aligned}$$

   are cells in R^{n+1}, and the graph Γ_f of f is a cell in R^n.

Fact 3.1.2 ([KPS86])**.** *Let $A_1, \ldots, A_n$ be definable subsets of R^n. Then there exists a partition of R^n into finitely many cells C_i* partitioning the *A_j, meaning that for all i, j either $C_i \subseteq A_j$ or $C_i \cap A_j = \varnothing$.*

Furthermore, for each definable function $f : A \to R$ with $A \subseteq R^n$, there is a partition of R^n into finitely many cells C_i partitioning A such that the restriction $f_{|C_i} : C_i \to R$ for each $C_i \subseteq A$ is continuous.

There are stronger versions of the cell decomposition theorem, which require the cells to be given by k-times differentiable functions, rather than continuous functions; see [VdD98, Section 7.3]. In the context of certain expansions of $\overline{\mathbb{R}}$ by analytic functions, there exists an even stronger version of the cell decomposition theorem that preserves analytic properties, see [vdDM94, Section 8]. We will only require the version applicable to $\mathbb{R}_{\text{exp}}$.

Definition 3.1.3. Over $\mathbb{R}_{\text{exp}}$, we define an *analytic cell* inductively:

1. Singletons and open intervals are analytic cells.

2. Let an analytic cell $C \subseteq \mathbb{R}^n$ be given and let $f, g : C \to \mathbb{R}$ be definable functions with an open neighbourhood U of C in $\mathbb{R}^n$ and real analytic functions $F, G : U \to R$ with $F_{|C} = f$ and $G_{|C} = g$. If $f < g$ on C, then

$$(f,g) := \{(x,r) \in C \times \mathbb{R} : f(x) < r < g(x)\},$$
$$(-\infty, f) := \{(x,r) \in C \times \mathbb{R} : r < f(x)\},$$
$$(f,+\infty) := \{(x,r) \in C \times \mathbb{R} : f(x) < r\}$$

are analytic cells in R^{n+1}, and the graph Γ_f of f is also an analytic cell in R^n.

Fact 3.1.4 (Analytic cell decomposition for $\mathbb{R}_{\text{exp}}$ [vdDM94, Theorem 8.8]). *Let $A_1, \ldots, A_n$ be definable subsets of $\mathbb{R}^n$. Then there exists a partition of $\mathbb{R}^n$ into finitely many analytic cells C_i partitioning the A_j as above.*

Furthermore, for each definable function $f : A \to \mathbb{R}$ with $A \subseteq R^n$, there is a partition of $\mathbb{R}^n$ into finitely many analytic cells C_i partitioning A such that the restriction $f_{|C_i} : C_i \to R$ for each $C_i \subseteq A$ is given by a real analytic function on a neighbourhood U_i of C_i.

3.2 Miller's Dichotomy

In [Mil94], Miller showed that in an o-minimal expansion $\mathcal{R}$ of $\overline{\mathbb{R}}$, either the real exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is definable or for every definable function $f : \mathbb{R} \to \mathbb{R}$ there exists an $n \in N$ and $a \in \mathbb{R}$ such that $f \leq x^n$ on (a, ∞). In the latter situation, we say that $\mathcal{R}$ is *polynomially bounded*. In [Mil96], Miller further generalized this result to arbitrary o-minimal structures expanding a real closed field.

Definition 3.2.1. A *power function* is an $\mathcal{L}(\mathcal{R})$-definable endomorphism of the ordered multiplicative group $R^{>} := \{x \in R : x > 0\}$.

Each power function f can be thought of as the function $x \mapsto x^{\lambda}$, where $\lambda := f'(1) \in \mathcal{R}$. The collection Λ of all such λ is a subfield of $\mathcal{R}$, called the field of exponents of $\mathcal{R}$. We say an o-minimal expansion $\mathcal{R}$ is powerbounded if every definable function is eventually bounded by a power function. By [Wil96, Proposition 4.2 and 4.6] in a power bounded $\mathcal{R}$ every power function is $\mathcal{L}(\varnothing)$-definable. Every model of T is power bounded with the same field of exponents Λ as $\mathcal{R}$. We then call T power bounded, and Λ the field of exponents of T.

Definition 3.2.2. An *exponential function* is an ordered group isomorphism $\exp : (R, +) \to (R^{>}, \cdot)$.

Fact 3.2.3 ([Mil96])**.** *Either $\mathcal{R}$ is power bounded or $\mathcal{R}$ defines an exponential function.*

By [Wil96, Section 2], if $\mathcal{R}$ defines an exponential function exp, then exp is $\mathcal{L}(\varnothing)$-definable.

Chapter 4

T-convexity and monomial groups

The concept of T-convex valuations was first studied by van den Dries and Lewenberg in [vdDL95]; it is a way to turn an o-minimal structure with theory T into a valued field that is compatible with the o-minimal structure. This chapter presents the collaborative work with Elliot Kaplan in [Kap23] on expansions of T-convex valued fields by a monomial group. In Section 4.1, we introduce the notion of T-convex valuations and present fundamental results in the study of power-bounded T-convex valued fields like the Wilkie inequality 4.1.5. Following this, we introduce the monomial group as a section of the valuation map in Section 4.2 and explore the structure theory of the power bounded T-convex valued fields with a monomial group. In particular, we establish a relative quantifier elimination result and various tameness properties. Section 4.2.2 concludes the chapter, where we establish that the theory of an exponential T-convex valued fields with a monomial group is not tame, by defining $\mathbb{N}$ in an archimedean model.

4.1 T-convex valuations on o-minimal structures

Let T be a complete o-minimal theory, extending the theory of real closed ordered fields in a language $\mathcal{L} \supseteq \mathcal{L}_{ring}$. Let $\mathcal{R}$ be a model of T.

Definition 4.1.1 ([vdDL95])**.** A *proper T-convex subring* is a proper convex subset $\mathcal{O} \subset \mathcal{R}$ which is closed under all $\mathcal{L}(\varnothing)$-definable continuous functions $f : R \to R$. Let $\mathcal{L}_{\mathcal{O}} := \mathcal{L} \cup \{\mathcal{O}\}$ and let $T_{\mathcal{O}}$ be the $\mathcal{L}_{\mathcal{O}}$-theory which extends T by axioms stating that $\mathcal{O}$ is a proper T-convex subring.

Let $(\mathcal{R}, \mathcal{O}) \vDash T_{\mathcal{O}}$. Then $\mathcal{O}$ is a convex subring of $\mathcal{R}$ (hence a valuation ring), and $(\mathcal{R}, \mathcal{O})$ is a convexly valued ordered field. As each element of the prime model $\mathcal{P}$ is $\mathcal{L}(\varnothing)$-definable, we always have $\mathcal{P} \subseteq \mathcal{O}$. We let $\Gamma := \mathcal{R}^\times / \mathcal{O}^\times$ denote the value group of $(\mathcal{R}, \mathcal{O})$, written additively, and we let $v : \mathcal{R}^\times \to \Gamma$ denote the surjective valuation map. If T is power bounded with field of exponents Λ, then Γ has the structure of an ordered Λ-vector space, where $\lambda \cdot v(a) := v(a^\lambda)$ for $\lambda \in \Lambda$ and $a \in \mathcal{R}^\times$, see [vdD97, Theorem B].

Fact 4.1.2 ([vdD97, Theorem 4.4])**.** *In the power bounded case, Γ is stably embedded as an ordered Λ-vector space in the language $\{\{\lambda x : \Gamma \to \Gamma\}_{\lambda \in \Lambda}, 0, +, <\}$.*

We write $\mathfrak{o}$ for the unique maximal ideal of $\mathcal{O}$, and we let $\boldsymbol{k} := \mathcal{O}/\mathfrak{o}$ denote the residue field of $(\mathcal{R}, \mathcal{O})$. We let $\pi : \mathcal{O} \to \boldsymbol{k}$ be the corresponding residue map; then π is order-preserving as $\mathcal{O}$ is convex. The residue field $\boldsymbol{k}$, considered with its induced structure, is a model of T; see [vdDL95, Remark 2.16] and [Yin17, Remark 2.3]. Moreover, $\boldsymbol{k}$ is stably embedded as a model of T [vdD97, Corollary 1.13].

4.1.1 Power bounded structures

For a set $A \subseteq R$, the downward closure is given as $A^{\downarrow} := \{x \in R : x < a \text{ for some } a \in A\}$. We say the cut of $A^{\downarrow}$ in $\mathcal{R}$ is realized by an element satisfying the partial type $p(x) = \{x > a : a \in A^{\downarrow}\} \cup \{x < c : c \notin A\}$.

Fact 4.1.3 ([vdDL95, Remark 3.8]). *Let $(\mathcal{R}, \mathcal{O}_{\mathcal{R}}) \models T_{\mathcal{O}}$ and let $\mathcal{S}$ be a simple T-extension of $\mathcal{R}$, i.e. generated by a single element. There are at most two T-convex valuation rings $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\mathcal{S}$ which make $\mathcal{S}$ a $T_{\mathcal{O}}$-extension of $\mathcal{R}$:*

$$\mathcal{O}_1 := \{y \in \mathcal{S} : |y| < u \text{ for some } u \in \mathcal{O}_{\mathcal{R}}\}, \quad \mathcal{O}_2 := \{y \in \mathcal{S} : |y| < d \text{ for all } d \in \mathcal{R} \text{ with } d > \mathcal{O}_{\mathcal{R}}\}$$

If the cut $\mathcal{O}_{\mathcal{R}}^{\downarrow}$ is realized by $b \in \mathcal{S}$, then b belongs to $\mathcal{O}_2$ but not $\mathcal{O}_1$, so $\mathcal{O}_1 \subsetneq \mathcal{O}_2$. If no element in $\mathcal{S}$ realizes the cut $\mathcal{O}_{\mathcal{R}}^{\downarrow}$, then $\mathcal{O}_1 = \mathcal{O}_2$.

Fact 4.1.4 ([Kap23, Corollary 1.2]). *Let $(\mathcal{R}, \mathcal{O}_{\mathcal{R}}) \models T_{\mathcal{O}}$ and let $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ be a simple $T_{\mathcal{O}}$-extension of $(\mathcal{R}, \mathcal{O}_{\mathcal{R}})$. If $\Gamma_{\mathcal{S}} = \Gamma_{\mathcal{R}}$, then*

$$\mathcal{O}_{\mathcal{S}} = \{y \in \mathcal{S} : |y| < d \text{ for all } d \in \mathcal{R} \text{ with } d > \mathcal{O}_{\mathcal{R}}\} = \mathcal{O}_2$$

If $\boldsymbol{k}_{\mathcal{S}} = \boldsymbol{k}_{\mathcal{R}}$, then

$$\mathcal{O}_{\mathcal{S}} = \{y \in \mathcal{S} : |y| < u \text{ for some } u \in \mathcal{O}_{\mathcal{R}}\} = \mathcal{O}_1$$

Fact 4.1.5 (The Wilkie Inequality [vdD97], Section 5). *Suppose that T is power bounded with field of exponents Λ. Let $(\mathcal{R}, \mathcal{O}_{\mathcal{R}}) \models T_{\mathcal{O}}$, let $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ be a $T_{\mathcal{O}}$-extension of $(\mathcal{R}, \mathcal{O}_{\mathcal{R}})$, and suppose that $\operatorname{rk}_{\mathcal{L}}(\mathcal{S} \mid \mathcal{R})$ is finite. Then*

$$\mathrm{rk}_{\mathcal{L}}(\mathcal{S} \mid \mathcal{R}) \geqslant \mathrm{rk}_{\mathcal{L}}(\boldsymbol{k}_{\mathcal{S}} \mid \boldsymbol{k}_{\mathcal{R}}) + \dim_{\Lambda}(\Gamma_{\mathcal{S}}/\Gamma_{\mathcal{R}}),$$

where $\dim_{\Lambda}$ *is the* Λ*-vector space dimension.*

Fact 4.1.6 ([Kap23, Corollary 1.4])**.** *Suppose that* T *is power bounded, let* $(\mathcal{R}, \mathcal{O}_{\mathcal{R}}) \models T_{\mathcal{O}}$*, and let* $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ *be a simple* $T_{\mathcal{O}}$*- extension of* $(\mathcal{R}, \mathcal{O}_{\mathcal{R}})$*. Then either* $\boldsymbol{k}_{\mathcal{S}} = \boldsymbol{k}_{\mathcal{R}}$ *or* $\Gamma_{\mathcal{S}} = \Gamma_{\mathcal{R}}$*.*

Of course, it may be the case that for an extension $(\mathcal{R}, \mathcal{O}_{\mathcal{R}}) \preccurlyeq (\mathcal{S}, \mathcal{O}_{\mathcal{S}}) \models T_{\mathcal{O}}$, we have that both $\boldsymbol{k}_{\mathcal{S}} = \boldsymbol{k}_{\mathcal{R}}$ and $\Gamma_{\mathcal{S}} = \Gamma_{\mathcal{R}}$. In this case, $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ is said to be an immediate extension of $(\mathcal{R}, \mathcal{O}_{\mathcal{R}})$.

Let (a_ρ) be a well-indexed sequence of elements of $\mathcal{R}$, i.e. ρ ranges over all ordinals less than λ for some limit ordinal λ. We say that (a_ρ) is *pseudocauchy* if there is an index ρ_0 such that

$$v(a_\tau - a_\sigma) > v(a_\sigma - a_\rho)$$

for all $\rho_0 < \rho < \sigma < \tau < \lambda$. An element a in some $T_{\mathcal{O}}$ extension of $\mathcal{R}$ is called a *pseudolimit* of (a_ρ) if there is an index ρ_0 such that

$$v(a - a_\sigma) > v(a - a_\rho)$$

for all $\rho_0 < \rho < \sigma < \lambda$. If $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ is an immediate extension of $(\mathcal{R}, \mathcal{O}_{\mathcal{R}})$, then for any $a \in \mathcal{S} \setminus \mathcal{R}$, there is a pseudocauchy sequence (a_ρ) in $\mathcal{R}$ with pseudolimit a and with no pseudolimits in $\mathcal{R}$; see [Kap42].

4.2 Expansions by a Monomial Group

The results in this section are part of a joint research project with Elliot Kaplan and were published in [KK24].

Definition 4.2.1. Let $(\mathcal{R}, \mathcal{O}) \models T_{\mathcal{O}}$. A section of v is a group homomorphism $s : \Gamma \to \mathcal{R}^{\times}$ such that $v \circ s$ is the identity on Γ. A monomial group for $(\mathcal{R}, \mathcal{O})$ is a multiplicative subgroup $\mathfrak{M} \subseteq \mathcal{R}^{\times}$ with $v : \mathfrak{M} \to \Gamma$ being a group isomorphism.

Let s be a section of v. Then the image $s(\Gamma)$ is a monomial group. Conversely, if $\mathfrak{M}$ is a monomial group, then $(v|_{\mathfrak{M}})^{-1}$ is a section of v. Any monomial group is necessarily a subgroup of $\mathcal{R}^{>}$, as $\mathcal{R}$ is real closed and Γ is divisible.

Definition 4.2.2. We say that a monomial group $\mathfrak{M} \subseteq \mathcal{R}^{>}$is T-compatible if either

- T is power bounded and $\mathfrak{M}$ is closed under all power functions, or
- T defines an exponential function exp and $\mathfrak{M}^{\succ} := \{\mathfrak{m} \in \mathfrak{M} : \mathfrak{m} > 1\}$ is closed under exp.

We say that a section s of v is T-compatible if the corresponding monomial group $s(\Gamma)$ is T-compatible. If $s : \Gamma \to \mathcal{R}^{>}$ is T-compatible and T is power bounded with field of exponents Λ, then s is an ordered Λ-vector space embedding.

Fact 4.2.3 ([Kap23, Lemma 1.5])**.** *Any model $(\mathcal{R}, \mathcal{O}) \models T_{\mathcal{O}}$ admits a T-compatible monomial group $\mathfrak{M}$.*

Definition 4.2.4. Let $\mathcal{L}_{\mathfrak{M}} := \mathcal{L}_{\mathcal{O}} \cup \{\mathfrak{M}\}$, and let $T_{\mathfrak{M}}$ be the $\mathcal{L}_{\mathfrak{M}}$-theory which extends $T_{\mathcal{O}}$ by axioms stating that $\mathfrak{M}$ is a T-compatible monomial group. Furthermore, for the sake of proving quantifier elimination, we will first be considering the three-sorted language with $\mathcal{L}_{\Gamma,k,s}$ consisting of an $\mathcal{L}$-sort for $\mathcal{R}$ and the residue field k each, a sort for the value group Γ, as well as the residue map $\pi : \mathcal{R} \to k$, the valuation $v : \mathcal{R} \to \Gamma$ and a section s of the valuation. Similarly, $\mathcal{L}_s$ is the expansion of $\mathcal{L}$ by just the section.

4.2.1 The power bounded case

In the following section, we will assume that T itself has quantifier elimination. This can be easily achieved by the process of skolemisation 3.0.1, which, due to definable choice, does not change the definable sets of the structure.

Theorem 4.2.5 ([KK24, Theorem 2.1])**.** *Suppose T eliminates quantifiers and has universal axiomatization. Then $T_{\mathfrak{M}}$ eliminates quantifiers in the language $\mathcal{L}_{\Gamma,k,s}$.*

Proof. We do the Schoenflies embedding test. Let $(\mathcal{R}, \Gamma_{\mathcal{R}}, k_{\mathcal{R}}) \vDash T_{\mathfrak{M}}$ and let $(\mathcal{S}, \Gamma_{\mathcal{S}}, k_{\mathcal{S}}) \vDash T_{\mathfrak{M}}$ be $|\mathcal{R}|^+$-saturated. Further let $(\mathcal{A}, \Gamma_{\mathcal{A}}, k_{\mathcal{A}})$ be a common $\mathcal{L}_{\Gamma,k,s}$-substructure of $\mathcal{R}$ and $\mathcal{S}$. For quantifier elimination of $T_{\mathfrak{M}}$, we need to show that the inclusion of $(\mathcal{A}, \Gamma_{\mathcal{A}}, k_{\mathcal{A}}) \subseteq (\mathcal{S}, \Gamma_{\mathcal{S}}, k_{\mathcal{S}})$ extends to an $\mathcal{L}_{\Gamma,k,s}$-embedding of $(\mathcal{R}, \Gamma_{\mathcal{R}}, k_{\mathcal{R}}) \to (\mathcal{S}, \Gamma_{\mathcal{S}}, k_{\mathcal{S}})$. Now, T has quantifier elimination and universal axiomatization by assumption, so $\mathcal{A}$ and $k_{\mathcal{A}}$ are both models of T and $\Gamma_{\mathcal{A}}$ is a Λ-vector space. By the existence of the section s, we already have that the valuation $v : \mathcal{A}^\times \to \Gamma_{\mathcal{A}}$ is surjective. The residue map $\pi : \mathcal{A} \to k_{\mathcal{A}}$, however, does not need to be surjective.

Claim 1. *As T eliminates quantifiers, by saturation we can already extend the inclusion $k_{\mathcal{A}} \subseteq k_{\mathcal{S}}$ to an $\mathcal{L}(k_{\mathcal{A}})$-embedding $k_{\mathcal{R}} \to k_{\mathcal{S}}$ on the residue field sort.*

Claim 2. *We can extend the inclusion $\Gamma_{\mathcal{A}} \subseteq \Gamma_{\mathcal{S}}$ to an embedding $\Gamma_{\mathcal{R}} \to \Gamma_{\mathcal{S}}$*

For this, assume we have an $\alpha \in \Gamma_{\mathcal{R}} \setminus \Gamma_{\mathcal{A}}$ that we want to embed into $\Gamma_{\mathcal{S}}$. Then by saturation, there is a $\beta \in \Gamma_{\mathcal{S}}$ realizing the same cut as α over $\Gamma_{\mathcal{A}}$. Taking monomials $\mathfrak{m} := s(\alpha)$ and $\mathfrak{n} := s(\beta)$, they realize the same cut over $\mathcal{A}$, so we get an $\mathcal{L}$-embedding $f : \mathcal{A}\langle\mathfrak{m}\rangle \to \mathcal{S}$ sending $\mathfrak{m} \mapsto \mathfrak{n}$. By 4.1.6 we have that $\pi(\mathcal{A}\langle m\rangle) = \pi(\mathcal{A}\langle\mathfrak{n}\rangle) = \pi(\mathcal{A})$ Then by 4.1.4 we have for any $y \in \mathcal{A}$:

$$v(y) \geq 0 \Leftrightarrow |y| < u \text{ for some } u \in \mathcal{A} \text{ with } v(u) \geq 0 \Leftrightarrow v(f(y)) \geq 0.$$

This makes f and $\mathcal{L}_{\mathcal{O}}$-embedding. For f to be an $\mathcal{L}_{\Gamma,k,s}$-embedding $(\mathcal{A}\langle\mathfrak{m}\rangle, \Gamma_{\mathcal{A}} \oplus \Lambda\alpha, k_{\mathcal{R}}) \to (\mathcal{S}, \Gamma_{\mathcal{S}}, k_{\mathcal{S}})$, note that for each $\gamma + \lambda\alpha \in \Gamma_{\mathcal{A}} \oplus \Lambda\alpha$, we have that

$$f(s(\gamma + \lambda\alpha)) = f(s(\gamma)\mathfrak{m}^{\lambda}) = s(\gamma)\mathfrak{n}^{\lambda} = s(\gamma + \lambda\beta).$$

Claim 3. *We can extend the inclusion $\mathcal{A} \subseteq \mathcal{S}$ such that $\pi : \mathcal{A} \to \boldsymbol{k}_{\mathcal{A}}$ is surjective.*

Suppose $\overline{a} \in \boldsymbol{k}_{\mathcal{A}} \backslash \pi(\mathcal{A})$. Let $a \in \mathcal{A}$ and $b \in \mathcal{S}$ be lifts of $\overline{a}$. Note that a and b both realize the cut

$$\{y \in \mathcal{A} : y < \mathcal{O}_{\mathcal{A}}\} \cup \{y \in \mathcal{O}_{\mathcal{A}} : \pi(y) < \overline{a}\}$$

so there is an $\mathcal{L}$-embedding $f : \mathcal{A}\langle a\rangle \to \mathcal{S}$ sending a to b. By Corollary 4.1.6, we have that $\Gamma_{\mathcal{A}} = \Gamma_{\mathcal{A}\langle a\rangle} = \Gamma_{\mathcal{A}\langle b\rangle}$. Then by Fact 4.1.4 we have for $y \in \mathcal{A}\langle a\rangle$ that

$$v(y) \geqslant 0 \Longleftrightarrow |y| < d \text{ for all } d \in \mathcal{A}^{>} \text{ with } v(d) < 0 \Longleftrightarrow v(f(y)) \geqslant 0.$$

Thus f is even an $\mathcal{L}_{\mathcal{O}}$-embedding, so it induces an $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-embedding $(\mathcal{A}\langle a\rangle, \Gamma_{\mathcal{A}}, \boldsymbol{k}_{\mathcal{R}}) \to (\mathcal{S}, \Gamma_{\mathcal{S}}, \boldsymbol{k}_{\mathcal{S}})$.

Now we extend the inclusion $(\mathcal{A}, \Gamma_{\mathcal{A}}, \boldsymbol{k}_{\mathcal{A}}) \subseteq (\mathcal{S}, \Gamma_{\mathcal{S}}, \boldsymbol{k}_{\mathcal{S}})$ to an $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-embedding $(\mathcal{R}, \Gamma_{\mathcal{R}}, \boldsymbol{k}_{\mathcal{R}}) \to (\mathcal{S}, \Gamma_{\mathcal{S}}, \boldsymbol{k}_{\mathcal{S}})$. By the previous claims, we can arrange that $\Gamma_{\mathcal{A}} = \Gamma_{\mathcal{R}}$ and $\pi(\mathcal{A}) = \boldsymbol{k}_{\mathcal{A}} = \boldsymbol{k}_{\mathcal{R}}$, so $\mathcal{R}$ is an immediate extension of $\mathcal{A}$. Let $a \in \mathcal{R} \setminus \mathcal{A}$, and take a pseudocauchy sequence (a_ρ) in $\mathcal{A}$ with pseudolimit a and with no pseudolimits in $\mathcal{A}$. Let $b \in \mathcal{S}$ be a pseudolimit of (a_ρ). Then by [Kap23, Corollary 2.11], there is a unique $\mathcal{L}_{\mathcal{O}}(\mathcal{A})$-embedding $f : \mathcal{A}\langle a \rangle \to \mathcal{S}$ sending a to b. This f induces an $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-embedding $(\mathcal{A}\langle a \rangle, \Gamma_{\mathcal{R}}, \boldsymbol{k}_{\mathcal{R}}) \to (\mathcal{S}, \Gamma_{\mathcal{S}}, \boldsymbol{k}_{\mathcal{S}})$. □

Corollary 4.2.6 ([KK24, Corollary 2.2])**.** *$T_{\mathfrak{M}}$ is complete and if T is model complete, then $T_{\mathfrak{M}}$ is model complete in the language $\mathcal{L}_{\mathfrak{M}}$.*

Proof. Let $\mathcal{P}$ be the prime model of T. Then $(\mathcal{P}, 0, \mathcal{P})$ admits an $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-embedding into any model of $T_{\mathfrak{M}}$ since $\boldsymbol{k} \models T$, so $T_{\mathfrak{M}}$ is complete; see [AvdHVdD17, Corollary B.11.7]. For model completeness in the language $\mathcal{L}_{\mathfrak{M}}$, let $\mathcal{R}$ and $\mathcal{S}$ be models of $T_{\mathfrak{M}}$ and assume that $\mathcal{S}$ is $|\mathcal{R}|^+$-saturated. Let $\mathcal{A} \models T_{\mathfrak{M}}$ be a common $\mathcal{L}_{\mathfrak{M}}$-substructure of $\mathcal{R}$ and $\mathcal{S}$. By a variant of Robinson's model completeness test [AvdHVdD17, Corollary B.10.4], it is enough to show that the inclusion $\mathcal{A} \subseteq \mathcal{S}$ extends to an embedding $\mathcal{R} \to \mathcal{S}$. As T is model complete, $T_{\mathcal{O}}$ is as well by [vdDL95, Corollary 3.13], so $\mathcal{A}$ is an elementary $\mathcal{L}_{\mathcal{O}}$-substructure of both $\mathcal{R}$ and $\mathcal{S}$.

Extending our language $\mathcal{L}$ by function symbols for $\mathcal{L}(\varnothing)$-definable functions, we arrange that T has quantifier elimination and universal axiomatization. Augmenting by additional sorts for the value group and residue field, we view $\mathcal{R}$ and $\mathcal{S}$ as $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-structures $(\mathcal{R}, \Gamma_{\mathcal{R}}, \boldsymbol{k}_{\mathcal{R}})$ and $(\mathcal{S}, \Gamma_{\mathcal{S}}, \boldsymbol{k}_{\mathcal{S}})$, where $s : \Gamma_{\mathcal{R}} \to \mathcal{R}^>$is the section corresponding to the monomial group $\mathfrak{M}$, and similarly for $\mathcal{S}$. Given $\gamma \in v(\mathcal{A}^\times)$, we have $v(s(\gamma)) = \gamma \in v(\mathcal{A}^\times)$, so $s(\gamma)$ belongs to $\mathfrak{M}_{\mathcal{A}}$. Thus, $(\mathcal{A}, v(\mathcal{A}^\times), \pi(\mathcal{A}))$ is a common $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-structure of $(\mathcal{R}, \Gamma_{\mathcal{R}}, \boldsymbol{k}_{\mathcal{R}})$ and $(\mathcal{S}, \Gamma_{\mathcal{S}}, \boldsymbol{k}_{\mathcal{S}})$. Theorem 4.2.5 gives an $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-embedding

$(\mathcal{R}, \Gamma_{\mathcal{R}}, \boldsymbol{k}_{\mathcal{R}}) \to (\mathcal{S}, \Gamma_{\mathcal{S}}, \boldsymbol{k}_{\mathcal{S}})$ over $(\mathcal{A}, \Gamma_{\mathcal{A}}, \boldsymbol{k}_{\mathcal{A}})$, which restricts to an $\mathcal{L}_{\mathfrak{M}}$-embedding $\mathcal{R} \to \mathcal{S}$ over $\mathcal{A}$. □

Corollary 4.2.7 ([KK24, Corollary 2.3]). *If the signature of $\mathcal{L}$ is finite and T is decidable, then $T_{\mathfrak{M}}$ is decidable as well.*

Corollary 4.2.8 ([KK24, Corollary 2.4]). *The value group Γ is purely stably embedded as an ordered Λ-vector space, and orthogonal to the residue field k, which is stably embedded as a model of T.*

Proof. By extending $\mathcal{L}$ by function symbols for all $\mathcal{L}(\varnothing)$-definable functions, we may assume that T has quantifier elimination and universal axiomatization. Let $(\mathcal{R}, \Gamma_{\mathcal{R}}, \boldsymbol{k}_{\mathcal{R}}) \models T_{\mathfrak{M}}$ and let $(\mathcal{A}, \Gamma_{\mathcal{A}}, \boldsymbol{k}_{\mathcal{A}})$ be an $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-substructure of $(\mathcal{R}, \Gamma_{\mathcal{R}}, \boldsymbol{k}_{\mathcal{R}})$. Let $\gamma, \gamma' \in \Gamma_{\mathcal{R}}^m$ with $\operatorname{tp}(\gamma/\Gamma_{\mathcal{A}}) = \operatorname{tp}(\gamma'/\Gamma_{\mathcal{A}})$ in the language of ordered Λ-vector spaces, and let $r, r' \in \boldsymbol{k}_{\mathcal{R}}^n$ with $\operatorname{tp}_{\mathcal{L}}(r/\boldsymbol{k}_{\mathcal{A}}) = \operatorname{tp}_{\mathcal{L}}(r'/\boldsymbol{k}_{\mathcal{A}})$. By a standard compactness argument, it suffices to show that (γ, r) has the same $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-type as (γ', r') over $(\mathcal{A}, \Gamma_{\mathcal{A}}, \boldsymbol{k}_{\mathcal{A}})$; see [CDH05]. As r and r' have the same type over $\boldsymbol{k}_{\mathcal{A}}$, we find an $\mathcal{L}$-isomorphism $\boldsymbol{k}_{\mathcal{A}}\langle r\rangle \to \boldsymbol{k}_{\mathcal{A}}\langle r'\rangle$ mapping r to r'. Using Claim 1 of the quantifier elimination proof, this extends to an $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-isomorphism

$$(\mathcal{A}, \Gamma_{\mathcal{A}}, \boldsymbol{k}_{\mathcal{A}}\langle r\rangle) \to (\mathcal{A}, \Gamma_{\mathcal{A}}, \boldsymbol{k}_{\mathcal{A}}\langle r'\rangle)$$

Let $\mathfrak{m}, \mathfrak{m}' \in \mathcal{R}^m$ be the tuples $(s(\gamma_1), \ldots, s(\gamma_m))$ and $(s(\gamma_1'), \ldots, s(\gamma_m'))$, respectively. Then $v(\mathcal{A}\langle\mathfrak{m}\rangle^\times) = \Gamma_{\mathcal{A}} \oplus \Lambda\gamma_1 \oplus \cdots \oplus \Lambda\gamma_m$, and similarly for $v\left(\mathcal{A}\langle\mathfrak{m}'\rangle^\times\right)$. By Claim 2 of the quantifier elimination proof, we get an $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-isomorphism

$$(\mathcal{A}\langle\mathfrak{m}\rangle, v(\mathcal{A}\langle\mathfrak{m}\rangle^\times), \boldsymbol{k}_{\mathcal{A}}\langle r\rangle) \to \left(\mathcal{A}\langle\mathfrak{m}'\rangle, v\left(\mathcal{A}\langle\mathfrak{m}'\rangle^\times\right), \boldsymbol{k}_{\mathcal{A}}\langle r'\rangle\right)$$

This isomorphism is elementary by our quantifier elimination, giving us stable embeddedness and orthogonality. Purity follows directly by taking our substructure to be $(\mathcal{P},\{0\},\mathcal{P})$, where $\mathcal{P}$ is the prime model of T. □

Corollary 4.2.9 ([KK24, Corollary 3.1])**.** *Suppose T eliminates quantifiers and has universal axiomatization. Then $T_{\mathfrak{M}}$ eliminates quantifiers in the language $\mathcal{L}_s$.*

Proof. Let $\mathcal{R}$ and $\mathcal{S}$ be models of $T_{\mathfrak{M}}$, and assume that $\mathcal{S}$ is $|\mathcal{R}|^{+}$-saturated. Let $\mathcal{A}$ be a common $\mathcal{L}_s$-substructure of $\mathcal{R}$ and $\mathcal{S}$. As in the proof of Corollary 4.2.6, we augment by additional sorts for the value group and residue field to get that $(\mathcal{A}, v(\mathcal{A}^\times), \pi(\mathcal{A}))$ is a common $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-substructure of $(\mathcal{R},\Gamma_{\mathcal{R}},\boldsymbol{k}_{\mathcal{R}})$ and $(\mathcal{S},\Gamma_{\mathcal{S}},\boldsymbol{k}_{\mathcal{S}})$, where the sections from the value group sort to the field sort are defined using the map s. Theorem 4.2.5 gives an $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$-embedding $(\mathcal{R},\Gamma_{\mathcal{R}},\boldsymbol{k}_{\mathcal{R}}) \to (\mathcal{S},\Gamma_{\mathcal{S}},\boldsymbol{k}_{\mathcal{S}})$ over $(\mathcal{A}, v(\mathcal{A}^\times), \pi(\mathcal{A}))$, which restricts to an $\mathcal{L}_s$-embedding $\mathcal{R}\to\mathcal{S}$ over $\mathcal{A}$. □

Lemma 4.2.10 ([KK24, Lemma 3.2])**.** *Let τ be a unary $\mathcal{L}_s(A)$-term. Then there are $m\in\mathbb{N}$, an $(m+1)$-ary $\mathcal{L}(A)$-definable function $f:\mathcal{R}^{m+1}\to\mathcal{R}$, and an $\mathcal{L}_s(A)$-definable set $B\subseteq\mathcal{R}^{m+1}$ such that:*

1. *The fibre of B over x, B_x, is open for each $x\in\mathcal{R}^m$, and $B_x\cap b_y=\varnothing$ for $x\neq y\in\mathcal{R}^m$.*
2. *$\mathcal{R}\setminus\left(\bigcup_{x\in\mathcal{R}^m}B_x\right)$ is a finite union of $\mathcal{L}_s(A)$-definable discrete sets.*
3. *For each $x\in\mathcal{R}^m$, we have $\tau(t)=f(x,t)$ for all $t\in B_x$.*

Proof. We proceed by induction on the complexity of terms. If τ is a variable or a constant symbol, then we take $m=0$, $f(t)=\tau(t)$, and $B=\mathcal{R}$ (here $\mathcal{R}^0$ is the one-point space).

Suppose that the lemma holds for all terms of lower complexity than τ. We first consider the case that $\tau = \sigma(\tau_1, \ldots, \tau_n)$ for $\mathcal{L}_s$-terms $\tau_1, \ldots, \tau_n$ and an $\mathcal{L}$-term σ. For each $i = 1, \ldots, n$, take $m_i \in \mathbb{N}$, an $\mathcal{L}(A)$ definable function $f_i : \mathcal{R}^{m_i+1} \to \mathcal{R}$, and an $\mathcal{L}_s(A)$-definable set $B_i \subseteq \mathcal{R}^{m_i+1}$ satisfying the conditions in
the lemma for τ_i. Let $m := m_1 + \ldots + m_n$ and define $B \subseteq \mathcal{R}^{m+1}$ and $f : \mathcal{R}^{m+1} \to \mathcal{R}$ as follows: for $x = (x_1, \ldots, x_n) \in \mathcal{R}^{m_1} \times \cdots \times \mathcal{R}^{m_n}$, put

$$B_x := B_{x_1} \cap \cdots \cap B_{x_n}, \quad f(x,t) = \sigma(f_1(x_1,t), \ldots, f_n(x_n,t))$$

Then m, B, and f satisfy the conditions in the lemma for τ.

Finally, suppose that $\tau = s(\sigma)$ for some $\mathcal{L}_s$-term σ. Take $m \in \mathbb{N}$ and an $\mathcal{L}(A)$-definable function $g : \mathcal{R}^{m+1} \to \mathcal{R}$, and an $\mathcal{L}_s(A)$-definable set $C \subseteq \mathcal{R}^{m+1}$ satisfying the conditions of the lemma for σ. As g is $\mathcal{L}(A)$-definable, there are $\mathcal{L}(A)$-definable functions $g_1, \ldots, g_k : \mathcal{R}^m \to \mathcal{R}$ such that $t \mapsto g(x,t) : \mathcal{R} \to \mathcal{R}$ is continuous on $\mathcal{R} \setminus \{g_1(x), \ldots, g_k(x)\}$ for all $x \in \mathcal{R}^m$. We let $C^* := C \setminus \bigcup_{i=1}^k \operatorname{Graph}(g_i)$. Note that then $t \mapsto g(x,t)$ is continuous on the open set $C_x^* = C_x \setminus \{g_1(x), \ldots g_k(x)\}$ for each x. Now we define $B \subseteq \mathcal{R}^{m+2}$ as follows: for $x \in \mathcal{R}^m$ and $y \in \mathcal{R}$, set

$$B_{x,y} := \begin{cases} \{t \in C_x^* : s(g(x,t)) = y\} & \text{if } y \in \mathfrak{M} \\ \operatorname{int}(\{t \in C_x^* : g(x,t) = 0\}) & \text{if } y = 0 \\ \varnothing & \text{otherwise.} \end{cases}$$

Then each $B_{x,y}$ is open, since $s^{-1}(y)$ is open for $y \in \mathfrak{M}$ and $t \mapsto g(x,t)$ is continuous on C_x^* for each x. Clearly, the sets $B_{x,y}$ are pairwise disjoint. Let $f : \mathcal{R}^{m+2} \to \mathcal{R}$ be given by $f(x,y,t) = y$. Then for $(x,y) \in \mathcal{R}^{m+1}$ and $t \in B_{x,y}$, we have

$$\tau(t) = s(\sigma(t)) = s(g(x,t)) = y = f(x,y,t)$$

It remains to show that $\mathcal{R}\backslash\bigcup_{x,y} B_{x,y}$ is a finite union of $\mathcal{L}_s(A)$-definable discrete sets. By assumption, $\mathcal{R}\backslash\bigcup_x C_x$ is a finite union of $\mathcal{L}_s(A)$-definable discrete sets, so it suffices to show that the $\mathcal{L}_s(A)$-definable set $\bigcup_x \left(C_x\backslash\bigcup_y B_{x,y}\right)$ is discrete. Since each C_x is open, it is enough to show that $C_x\backslash\bigcup_y B_{x,y}$ is finite for each x, and this holds since $C_x\backslash\bigcup_y B_{x,y}$ is contained in union of $\{g_1(x),\dots,g_k(x)\}$ and the boundary of the set $\{t \in \mathcal{R} : g(x,t) = 0\}$. □

Theorem 4.2.11 ([KK24, Theorem 3.3])**.** *Every $\mathcal{L}_s(A)$-definable subset of $\mathcal{R}$ is the union of an $\mathcal{L}_s(A)$-definable open set and finitely many $\mathcal{L}_s(A)$-definable discrete sets.*

Proof. Let $D \subseteq \mathcal{R}$ be $\mathcal{L}_s(A)$-definable. By removing the interior of D (which is open and $\mathcal{L}_s(A)$-definable), we may assume that D has empty interior. We will show that D is a finite union of $\mathcal{L}_s(A)$-definable discrete sets. By quantifier elimination, we may assume that D is of the form

$$D = \{t \in \mathcal{R} : \tau_0(t) = 0, \tau_1(t) < 0, \dots, \tau_n(t) < 0\}$$

for $\mathcal{L}_s$-terms $\tau 0,\dots,\tau_n$. For each $i \leqslant n$, take $m_i \in \mathbb{N}$, an $\mathcal{L}(A)$-definable function $f_i : \mathcal{R}^{m_i+1} \to \mathcal{R}$, and an $\mathcal{L}_s(A)$-definable set $B_i \subseteq \mathcal{R}^{m_i+1}$ as in Lemma 4.2.10. Let $m := m_0 + \dots + m_n$ and for $x = (x_0,\dots,x_n) \in \mathcal{R}^{m_0} \times \dots \times \mathcal{R}^{m_n}$, set $B_x := B_{x_0} \cap \dots \cap B_{x_n}$. Then each B_x is open, $\mathcal{R}\backslash\bigcup_{x\in\mathcal{R}^m} B_x$ is a finite union of $\mathcal{L}_s(A)$-definable discrete sets, and for each x, we have

$$D \cap B_x = \{t \in B_x : f_0(x_0,t) = 0, f_1(x_1,t) < 0, \dots, f_n(x_n,t) < 0\}$$

As each f_i is $\mathcal{L}(A)$-definable and B_x is open, we see that $D \cap B_x$ is finite (otherwise, $D \cap B_x$ has interior). Thus, $D \cap \bigcup_{x \in \mathcal{R}^m} B_x$ is $\mathcal{L}_s(A)$-definable and discrete. As $\mathcal{R} \setminus \bigcup_{x \in \mathcal{R}^m} B_x$ is a finite union of $\mathcal{L}_s(A)$-definable discrete sets, we conclude that D is a finite union of $\mathcal{L}_s(A)$-definable discrete sets. □

Hieronymi and Nell introduced a criterion for establishing distality in language extensions of o-minimal structures [HN17, Theorem 2.1] extending ordered abelian groups. We will directly state its requirements in the context of our setting during the proof of the following theorem.

Theorem 4.2.12 ([KK24, Theorem 4.1])**.** *$T_{\mathfrak{M}}$ is distal.*

Proof. Let $(\mathcal{U}, \mathcal{O}_{\mathcal{U}}, \mathfrak{M}_{\mathcal{U}}) \models T_{\mathfrak{M}}$ be a monster model. As in the previous section, we assume that T has quantifier elimination and universal axiomatization, and we work in the language $\mathcal{L}_s$, so $s(\mathcal{U}^\times) = \mathfrak{M}_{\mathcal{U}}$. We will use the Hieronymi-Nell criterion for distality [HN17, Theorem 2.1], applied to our theory T with additional function symbol s. In our case, we need to verify the following:

1. The theory $T_{\mathfrak{M}}$ has quantifier elimination in the language $\mathcal{L}_s$.

2. For every $\mathcal{L}_s$-substructure $\mathcal{R} \subseteq \mathcal{U}$ and every $c \in \mathcal{U}^m$, there is a tuple $d \in s(\mathcal{R}\langle c\rangle)^n$ for some n such that $s(\mathcal{R}\langle c\rangle) \subseteq \langle s(\mathcal{R}), d\rangle$.

3. Suppose that $k' \leqslant k$ and g, h are $\mathcal{L}$-terms of arities $k+m$ and $k'+n$ respectively, $b_1 \in \mathcal{U}^m$, and $b_2 \in \mathfrak{M}_{\mathcal{U}}^n$. If $(a_i)_{i \in I}$ is an $\mathcal{L}_s(\varnothing)$-indiscernible sequence from $\mathfrak{M}_{\mathcal{U}}^{k'} \times \mathcal{U}^{k-k'}$ such that

   (a) $I = I_1 + c + I_2$, where I_1 and I_2 are infinite without endpoints, and $(a_i)_{i \in I_1 + I_2}$ is $\mathcal{L}_s(b_1 b_2)$ indiscernible, and

(b) $s(g(a_i, b_1)) = h(a_i, b_2)$ for every $i \in I_1 + I_2$, then $s(g(a_c, b_1)) = h(a_c, b_2)$.

We have already verified (1) in Corollary 4.2.9 above. For (2), let $\mathcal{R}$ be an $\mathcal{L}_s$-substructure of $\mathcal{U}$ and let $c \in \mathcal{U}^m$. Then $s(\mathcal{R}\langle c\rangle)$ is a finitely generated multiplicative Λ-vector space over $s(\mathcal{R})$ by the Wilkie Inequality 4.1.5. Take generators $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \in s(\mathcal{R}\langle c\rangle)$. Then

$$s(\mathcal{R}\langle c\rangle) \subseteq \langle s(\mathcal{R}), \mathfrak{m}_1, \ldots, \mathfrak{m}_n\rangle$$

Finally, let $f, g, (a_i), b_1, b_2$ be as in (3), so $s(g(a_i, b_1)) = h(a_i, b_2)$ for every $i \in I_1 + I_2$. We may as well assume that $g(a_i, b_1)$ and $h(a_i, b_2)$ are nonzero for these i (otherwise, $s(g(a_c, b_1)) = h(a_c, b_2) = 0$ as well, since T is distal). We first claim that $h(a_i, b_2) \in \mathfrak{M}_{\mathcal{U}}$ for all $i \in I$. Fix $i \in I_1 + I_2$, so $h(a_i, b_2) = h(a_{i,1}, \ldots, a_{i,k'}, b_2) \in \mathfrak{M}_{\mathcal{U}}$. Let $\mathcal{R}$ be the $\mathcal{L}$-substructure of $\mathcal{U}$ generated by $(a_{i,1}, \ldots, a_{i,k'}, b_2)$. Since $(a_{i,1}, \ldots, a_{i,k'}, b_2) \in \mathfrak{M}_{\mathcal{U}}^{k'+n}$, the Wilkie Inequality 4.1.5 tells us that $s(\mathcal{R})$ is the multiplicative Λ-vector space generated by $(a_{i,1}, \ldots, a_{i,k'}, b_2)$, so in particular

$$h(a_i, b_2) = a_{i,1}^{\lambda_1} \cdots a_{i,k'}^{\lambda_{k'}} b_2^{\lambda} \tag{*}$$

for some $\lambda_1, \ldots, \lambda_{k'} \in \Lambda$ and some tuple $\lambda \in \Lambda^n$. Since T is distal, the equality (*) holds for all $i \in I$. Thus $h(a_c, b_2)$ is a product of Λ-powers of elements in $\mathfrak{M}_{\mathcal{U}}$, so $h(a_c, b_2) \in \mathfrak{M}_{\mathcal{U}}$. Therefore, in order to show that $s(g(a_c, b_1)) = h(a_c, b_2)$, it is enough to show that $v(g(a_c, b_1)) = v(h(a_c, b_2))$. This holds since $v(g(a_i, b_1)) = v(h(a_i, b_2))$ for all $i \in I_1 + I_2$ and since $T_{\mathcal{O}}$ is distal, as $T_{\mathcal{O}}$ is weakly o-minimal, and thus by [DGL11, Theorem 4.1] dp-minimal, hence by [Sim13, Lemma 2.10] distal. □

Proposition 4.2.13 ([KK24, Proposition 4.2]). *$T_{\mathfrak{M}}$ has NIP, however, $T_{\mathfrak{M}}$ defines an infinite discrete set, hence it is not strongly dependent.*

Proof. All distal theories are dependent by definition. To see that $T_{\mathfrak{M}}$ is not strongly dependent, let $(\mathcal{U}, \mathcal{O}_{\mathcal{U}}, \mathfrak{M}_{\mathcal{U}}) \models T_{\mathfrak{M}}$ be sufficiently saturated, and note that for each $\varepsilon \in \mathcal{U}^{>0}$, the set $\mathfrak{M}_{\mathcal{U}} \cap (0, \varepsilon)$ is definable, discrete, and infinite. By [DG17, Theorem 2.11] $T_{\mathfrak{M}}$ is not strongly dependent. □

4.2.2 The exponential case

For this section, we assume T is o-minimal. Further, we assume that an exponential function $\exp : R \to R$ is definable in $\mathcal{R}$ and $\mathfrak{M}^{\succ}$ is closed under exp.

Lemma 4.2.14 ([KK24, Lemma 5.1])**.** *The additive group of $\mathcal{R}$ admits a direct sum decomposition $\mathcal{R} = \mathcal{O} \oplus \log(\mathfrak{M})$.*

Proof. We will conclude the statement from the claim that $\exp(\mathcal{O}) = (\mathcal{O}^{\times})^{>}$. For the first direction, take $a \in \mathcal{O}$. Then by T-convexity, both $\exp a$ and $\frac{1}{\exp a} = \exp(-a)$ belong to $\mathcal{O}^{>}$, as exp is $\mathcal{L}(\varnothing)$-definable. Thus $a \in (\mathcal{O}^{\times})^{>}$. For the other direction, take $u \in (\mathcal{O}^{\times})^{>}$. In the case that $u \geq 1$, we have that $\log u \in \mathcal{O}$ as $\mathcal{O}$ is convex and $0 \leq \log u < u$. If $u < 1$, then the previous argument works for $\frac{1}{u} > 1$ and so $\log(\frac{1}{u}) \in \mathcal{O}$ as before. So $\log u = -\log \frac{1}{u} \in \mathcal{O}$ as well. Now that we have shown that $\exp(\mathcal{O}) = (\mathcal{O}^{\times})^{>}$, consider that $\mathcal{R}^{>} = (\mathcal{O}^{\times})^{>} \oplus \mathfrak{M}$ as an internal direct sum of the multiplicative groups. Applying log yields the statement. □

We now follow the strategy Camacho used for showing that Hahn fields with a predicate for the subring of purely infinite elements are undecidable [CA18, Section 4.2]. For this, take $a \in \mathcal{R}$ and $\mathfrak{m} \in \mathfrak{M}$. Lemma 4.2.14 gives us a unique $b \in \mathfrak{m}\log(\mathfrak{M})$

with $a - b \in \mathfrak{m}\mathcal{O}$. We define $a|\mathfrak{m}$ to be this element b, so $(a, \mathfrak{m}) \mapsto a|\mathfrak{m}$ is an $\mathcal{L}_{\mathfrak{M}}(\varnothing)$-definable function. We also define

$$\operatorname{supp}(a) := \{\mathfrak{m} \in \mathfrak{M} : s\,(a - a|_{\mathfrak{m}}) = \mathfrak{m}\}\,.$$

Then $\operatorname{supp}(a)$ is an $\mathcal{L}_{\mathfrak{M}}(a)$-definable subset of $\mathfrak{M}$. The element $a|\mathfrak{m}$ can be thought of as a "truncation of a at $\mathfrak{m}$," and $\operatorname{supp}(a)$ serves as an analog of the support. Indeed, viewing the example of transseries $\mathbb{T}$ as a model of $T_{\text{an, exp}}$, the element $a|_{\mathfrak{m}}$ is exactly the truncation of an element $a \in \mathbb{T}$ at a transmonomial $\mathfrak{m}$, and the set $\operatorname{supp}(a)$ is exactly the support of a. If $\mathfrak{m}$ is an infinitesimal transmonomial in $\mathbb{T}$, then the support of $(1 - \mathfrak{m})^{-1}$ is the set $\{\mathfrak{m}^n : n \in \mathbb{N}\}$. Thus, $\mathbb{N} = \{\log \mathfrak{n} / \log \mathfrak{m} : \mathfrak{n} \in \operatorname{supp}(1 - \mathfrak{m})^{-1}\}$ is definable in $\mathbb{T}$. We will show that a suitable analogue holds in our model $\mathcal{R}$.

Proposition 4.2.15 ([KK24, Proposition 5.2])**.** *Let $\mathfrak{m} \in \mathfrak{M}$ with $\mathfrak{m} < 1$. Then for all $n \in \mathbb{N}m$ $\mathfrak{m}^n \in \operatorname{supp}(\frac{1}{1-\mathfrak{m}})$, and if $\mathfrak{n} \in \operatorname{supp}(\frac{1}{1-\mathfrak{m}})$, then $\mathfrak{n} = \mathfrak{m}^n$ for some $n \in \mathbb{N}$ or $\mathfrak{n} < \mathfrak{m}^n$ for all $n \in \mathbb{N}$.*

Proof. As $\mathfrak{m} \in \mathfrak{M}$ is < 1, we have that $\mathfrak{m} \in \mathfrak{o}$, the maximal ideal of $\mathcal{O}$. So both $1 - \mathfrak{m}$ and $\frac{1}{1-\mathfrak{m}}$ belong to $\mathcal{O}^{\times}$. As we have that $\frac{1}{1-\mathfrak{m}}|1 = 0$, we have that $1 \in \operatorname{supp}\frac{1}{1-\mathfrak{m}}$. Further if $\mathfrak{n} \in \operatorname{supp}\frac{1}{1-\mathfrak{m}}$, then $\mathfrak{n} \leq 1$. Now fix an $n \in \mathbb{N}$ and take an $\mathfrak{n} \in \mathfrak{M}$ satisfying $\mathfrak{m}^{n+1} \leq \mathfrak{n} < \mathfrak{m}$. We now want to show that $\mathfrak{n} \in \operatorname{supp}\frac{1}{1-\mathfrak{m}}$ if and only if $\mathfrak{n} = \mathfrak{m}^{n+1}$. As

$$1 + \mathfrak{m} + \cdots + \mathfrak{m}^n = \mathfrak{n}\left(\frac{1}{\mathfrak{n}} + \cdots + \frac{\mathfrak{m}^n}{\mathfrak{n}}\right),$$

and $\frac{1}{\mathfrak{n}}, \ldots, \frac{\mathfrak{m}^n}{\mathfrak{n}} \in \mathfrak{M}^{\succ} \subseteq \log(\mathfrak{M})$, we have that their sum is in $\log(\mathfrak{M})$ as well, so $(1 +$

$\mathfrak{m} + \cdots + \mathfrak{m}^n) \in \mathfrak{n}\log(\mathfrak{M})$. Then

$$\frac{1}{1-\mathfrak{m}} - 1 - \mathfrak{m} - \cdots - \mathfrak{m}^n = \frac{\mathfrak{m}^{n+1}}{1-\mathfrak{m}} \in \mathfrak{n}\mathcal{O},$$

so the truncation $\frac{1}{1-\mathfrak{m}}|\mathfrak{n} = 1 + \mathfrak{m} + \cdots + \mathfrak{m}^n$. Then the section $s(\frac{1}{1-\mathfrak{m}} - (\frac{1}{1-\mathfrak{m}}|_{\mathfrak{n}})) = \mathfrak{m}^{n+1}$, so it follows that $\mathfrak{n} \in \operatorname{supp}(\frac{1}{1-\mathfrak{m}})$ if and only if $\mathfrak{n} = \mathfrak{m}^{n+1}$. □

Corollary 4.2.16 ([KK24, Corollary 5.3])**.** *There is a definable set $A \subseteq \mathcal{R}$ with $\mathbb{N} \subseteq A$ and if $a \in A \setminus \mathbb{N}$, then $a > \mathbb{N}$. That means $\mathbb{N}$ is externally definable in any model of $T_{\mathfrak{M}}$ and if $T_{\mathfrak{M}}$ has an archimedean model, then $\mathbb{N}$ is definable in a model of $T_{\mathfrak{M}}$.*

Proof. Take a monomial $\mathfrak{m} \in \mathfrak{M}$ with $\mathfrak{m} < 1$ and define

$$A := \left\{ a \in \mathcal{R} : \exp(a \log \mathfrak{m}) \in \operatorname{supp}\left(\frac{1}{1-\mathfrak{m}}\right)\right\}.$$

By Proposition 4.2.15, we have for each $a \in A$ either $a \in \mathbb{N}$ or $a > \mathbb{N}$. Hence $\mathbb{N}$ is externally definable as the intersection of A and a convex subset of $\mathcal{R}$. Now assume that T has an archimedean model, then there is a $\mathcal{S} \models T_{\mathfrak{M}}$ with valuation ring $\mathcal{O}_{\mathcal{S}} = a \in \mathcal{S} : |a| < n$ for some $n \in \mathbb{N}$. Defining A as above, we have in $\mathcal{S}$ that $\mathbb{N} = A \cap \mathcal{O}_{\mathcal{S}}$. □

This leads to $T_{\mathfrak{M}}$ inheriting the stability-theoretic properties that Peano arithmetic has. We immediately have that

- $T_{\mathfrak{M}}$ is undecidable, by [Gö31].
- $T_{\mathfrak{M}}$ is dependent, and therefore not distal.

Further, we can show:

Lemma 4.2.17 ([KK24, Corollary 5.4])**.** *Let T be an o-minimal theory defining* exp *and $T_{\mathfrak{M}}$ the corresponding theory of a T-convex valued field with a predicate for a monomial group. Then any completion of $T_{\mathfrak{M}}$ has ATP.*

Proof. We follow the argument in [AKL23, Example 4.31]. First, assume that our structure $\mathcal{R} \models T_{\mathfrak{M}}$ is sufficiently saturated and let A be the definable set from Corollary 4.2.16 satisfying $\mathbb{N} \subseteq A$ and for all $a \in A$ either $a \in \mathbb{N}$ or $a > \mathbb{N}$. Then let $\phi(x,y)$ be the formula capturing division on $\mathbb{N}$ in the form that $x \in A \setminus \{1\}$ and $x \cdot z = y$ for some $z \in A$. Now we need to find a tuple of parameters $(a_\eta)_{\eta \in 2^{<\omega}}$ such that $\{\phi(x,a_\eta) : \eta \in I\}$ is consistent for $I \subseteq 2^{<\omega}$ if and only if I is an antichain. By saturation and compactness, it is sufficient to find a tuple of parameters $(a_\eta)2^{<n}$ such that $\{\phi(x,a_\eta) : \eta \in I\}$ if and only if $I \subseteq 2^{<\omega}$ is an antichain for each $n < \omega$. So let one n, ω be given. Then we can enumerate all antichains $I_1, \dots, I_m$ in $2^{<n}$ and let $p_1, \dots, p_m$ be the first m prime numbers. Then assigning for each $\eta \in 2^{<n}$, $a_\eta = \prod\{p_i : \eta \in I_i\}$. Then let $I \subseteq 2^{<n}$ be an arbitrary subset. If I is an antichain, then $I = I_j$ for some $1 \leq j \leq m$ and the consistency of $\{\phi(x,a_\eta) : \eta \in I\}$ gets witnessed by p_j. For the other direction, suppose the consistency of $\{\phi(x,a_\eta) : \eta \in I\}$ is witnessed by some $b \in A \setminus 1$. Then $b \leq a_\eta$ for each $\eta \in I$ so $b \in \mathbb{N}$, since all $a_\eta \in \mathbb{N}$. Now any prime factor p of b is also a prime factor of any a_η for $\eta \in I$, so $p = p_j$ for some $1 \leq j \leq m$, so $I \subseteq I_j$, so I is an antichain. ☐

Chapter 5

The j-function and pfaffian chains

The goal of this chapter is to present the work in [Kes23], showing that the j-function is not $\mathbb{R}_{\text{exp}}$-pfaffian. First, we introduce our main protagonist, the modular j-function in Section 5.1. The function j satisfies a differential equation, so we give an expository account of Freitag and Scanlon's proof of the strong minimality of said differential equation from [FS18], Theorem see 5.1.9, including Nishioka's contribution [Nis89] to the study of the j-function. After establishing the strong minimality, we focus on pfaffian chains in Section 5.2.1. First, define pfaffian chains and present Freitag's result [Fre21] on j being not $\mathbb{C}$-pfaffian, see Theorem 5.2.3, using the strong minimality of j's differential equation. In subsection 5.2.2, we define the restriction of $\mathsf{j} = j(ix) : (1, \infty) \to \mathbb{R}$ and establish that j is not $\overline{\mathbb{R}}$-pfaffian following Freitags approach, see Theorem 5.2.10. Lastly, in subsection 5.2.3, we establish that that j is also not $\mathbb{R}_{\text{exp}}$-pfaffian, see 5.2.16, using the strong minimality of j and an Ax-Schanuel theorem from [BSCFN23].

5.1 The j-function and Nishiokas results

The j-function is an important modular function $j : \mathbb{H} \to \mathbb{C}$ with $\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}$ being the upper half plane. The j-function is an analytic function on $\mathbb{H}$ with Fourier expansion at $t \in \mathbb{H}$ given by

$$j(t) = \exp(-2\pi it) + 744 + \cdots + \text{higher order terms in } e^{2\pi it}$$

The j-function is a modular function for $SL_2(\mathbb{Z})$, which means that for $T \in SL_2(\mathbb{Z})$ and all $t \in \mathbb{H}$, we have $j(Tt) = j(t)$, where the action is given by *Möbius transformations*:

$$Tt = \frac{at+b}{ct+d} \text{ for } T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We call a function f being invariant under the action of a subgroup G of $SL_2(\mathbb{C})$ G-automorphic. Moreover, for $a, b \in \mathbb{H}$, we only have $j(a) = j(b)$ if there is a $T \in SL_2(\mathbb{Z})$ such that $a = T \cdot b$.

This means that the entire behaviour of j can be represented on a domain V whose images under the action of $SL_2(\mathbb{C})$ partition the upper half plane. Such a domain V is called a *fundamental domain*, and j maps V bijectively onto $\mathbb{C}$. We will oftentimes be considering j on a neighbourhood $U \subseteq V$. Below is a picture of a standard choice $V = \left\{\tau \in \mathbb{H} \,\middle|\, |\tau| > 1, \ -\frac{1}{2} \leq Re(\tau) < \frac{1}{2}\right\} \ \cup \ \left\{\tau \in \mathbb{H} \,\middle|\, |\tau| = 1, \ -\frac{1}{2} \leq Re(\tau) \leq 0\right\}$ fundamental domain for j,

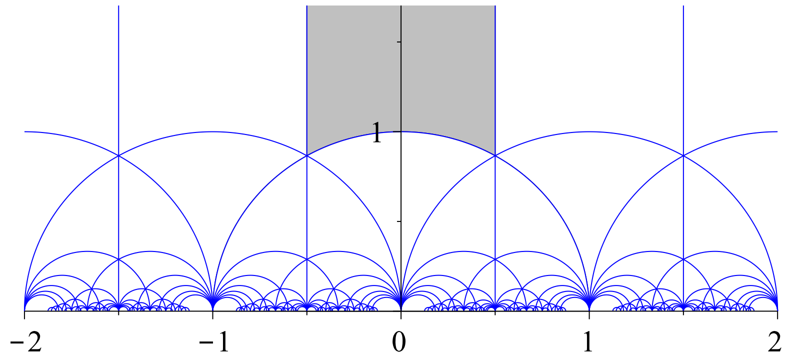

Figure 5.1: Fundamental domain for the modular group on the upper half-plane. Image by Kilom691 & Hulpke (2017), CC BY-SA 4.0, Wikimedia Commons [Kil17].

Further, j satisfies an order 3 algebraic differential equation, namely

$$\chi(y) := \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2 + (y')^2 \cdot \frac{y^2 - 1968y + 2654208}{y^2(y-1728)} = 0.$$

This equation splits into a Schwarzian defined as

$$S(y) = \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2$$

and a rational term

$$R(y) = \frac{y^2 - 1968y + 2654208}{y^2(y-1728)}$$

rewriting the equation as

$$\chi(y) = S(y) + (y')^2 R(y).$$

Remark 5.1.1. *This Schwarzian satisfies a chain rule $S(f \circ g) = (g')^2 S(f) \circ g + S(g)$, and two functions f and g satisfy $S(f) = S(g)$ if and only if $f = \frac{ag+b}{cg+d}$ for constants a, b, c, d satisfying $ad - bd \neq 0$, meaning S is also invariant under Möbius transformations. It is worth noting that if f' is constant, then $S(f) = 0$, so the* *functions g satisfying $S(g) = 0$ are precisely the degree one rational functions over the field of constants.*

Our preferred differential field for the study of the differential equation is the field of meromorphic functions $\mathcal{M}(U)$ on a nonempty subset U of the fundamental domain of j.

Following a conjecture by Mahler [Mah69], Nisioka proved the following:

Theorem 5.1.2 ([Nis89])**.** *Let G be a Zariski dense subgroup of $SL_2(\mathbb{C})$. Then no nontrivial G-automorphic function satisfies an algebraic differential equation of order 2 or less over $\mathbb{C}$.*

This applies to j, as $SL_2(\mathbb{Z})$ is Zariski dense in $SL_2(\mathbb{C})$. Below, we give a sketch of the proof:

Lemma 5.1.3 ([Nis89])**.** *Suppose a subgroup G of $SL_2(\mathbb{C})$ is Zariski dense in $SL_2(\mathbb{C})$. Let $M := \mathcal{M}(U)$ be a field of meromorphic functions on a non-empty domain U in $\mathbb{C}$ which contains the rational function field $\mathbb{C}(z)$ and which is stable under differentiation $\frac{d}{dz}$. Let P be a polynomial in $M[X, Y, Z]$ with*

$$P\left(Tz, \frac{d}{dz}Tz, \frac{d^2}{dz^2}Tz\right) = 0 \text{ for all } T \in G.$$

Then P is identically 0.

Proof. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the equation written out reads as:

$$P\left(\frac{az+b}{cz+d}, \frac{1}{(cz+d)^2}, \frac{1}{-2c(cz+d)^3}\right) = 0 \tag{A}$$

Treating $SL_2(\mathbb{C})$ as a variety let $K = \mathbb{C}(SL_2(\mathbb{C}))$ be the rational function field on $SL_2(\mathbb{C})$ equipped with $\frac{d}{dz}K = 0$. Then K and M are linearly disjoint field extensions over $\mathbb{C}$ as K extends the differential constants and M extends to functions with non-zero derivative. So there is no algebraic relation between these two extensions, and we have that $trdeg MK/M = 3$, as the dimension of $SL_2(\mathbb{C})$ as a variety is 3. Let

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a generic point of $SL_2(\mathbb{C})$ i.e. of maximal transcence degree within in K.

If $P \neq 0$, we have an algebraic dependence of the variables in equation (A) $\frac{az+b}{cz+d}, \frac{1}{(cz+d)^2}, \frac{1}{-2c(cz+d)^3}$ over $\mathbb{C}$ Then $az+b, c, d$ are dependent over M, but $z \in M \supseteq \mathbb{C}(z)$, so $az + b$ is algebraic over $M(c, d)$. By differentiating the dependence relation, a is algebraic over $M(c, d)$. Contradiction. □

Now, following corrections from [FS18] of Nishioka's original proof, we can give the proof of Nishioka's main theorem above.

Proof of Nishioka's Theorem 5.1.2. Let $f(t)$ be G-automorphic function. Assume for a contradiction that $t, f(t), \frac{d}{dt}f(t), \frac{d^2}{dt^2}f(t)$ are algebraically dependent over $\mathbb{C}$. Then there is a polynomial P over $\mathbb{C}$ with

$$P\left(t, f(t), \frac{d}{dt}f(t), \frac{d^2}{dt^2}f(t)\right) = 0.$$

Now, as f is G-automorphic, we may replace t with Tt for each $T \in G$. This substitution yields the variables plugged into P:

1. Tt

2. $f(Tt) = f(t)$

3. $\frac{d}{dTt} f(Tt) = \left(\frac{d}{dt}Tt\right)^{-1} \frac{d}{dt} f(t)$

4. $\left(\frac{d}{dTt}\right)^2 f(Tt) = \left(\frac{d}{dt}Tt\right)^{-2} \frac{d^2}{dt^2} f(t) - \left(\frac{d}{dt}Tt\right)^{-3} \left(\frac{d^2}{dl^2}Tt\right) \frac{d}{dt} f(t)$.

Substituting that, we have:

$$P\left(Tt, f(t), \left(\frac{d}{dt}Tt\right)^{-1} \frac{d}{dt} f(t), \left(\frac{d}{dt}Tt\right)^{-2} \frac{d^2}{dt^2} f(t) - \left(\frac{d}{dt}Tt\right)^{-3} \left(\frac{d^2}{dt^2}Tt\right) \frac{d}{dt} f(t)\right) = 0.$$

By clearing denominators, we obtain a nonzero polynomial over $\mathbb{C}(t)$ which vanishes on $\left(Tt, \frac{d}{dt}Tt, \frac{d^2}{dt^2}Tt\right)$ for all $T \in G$. This contradicts Lemma 5.1.3. $\square$

5.1.1 The strong minimality of $\chi = 0$

In this section, we will present the proof by Freitag and Scanlon from [FS18] that the set defined by the differential equation $\chi(y) = 0$, which j satisfies, is strongly minimal in DCF_0. For this, we need two more ingredients. First, we need the Seidenberg Embedding Theorem:

Fact 5.1.4 ([Sei58])**.** *Let $K = \mathbb{Q}\langle u_1, \ldots, u_n\rangle$ be the differential field generated in $\mathcal{U}$ by n elements over $\mathbb{Q}$ and let $K_1 = K\langle v\rangle$ be a simple differential field extension of K (i.e. generated by one element as a differential field in $\mathcal{U}$). Let $U \subset \mathbb{C}$ be an open ball and $\iota : K \to \mathcal{M}(U)$ a differential field embedding of K into the differential field of meromorphic functions on U. Then there is an open ball $V \subseteq U$ and an extension of ι to a differential field embedding of K_1 into $\mathcal{M}(V)$.*

The second fact we need is a consequence of Pila's modular Ax-Lindemann-Weierstrass Theorem with derivatives.

Fact 5.1.5 ([Pil13a])**.** *Suppose that $a_1, \dots, a_n \in \mathbb{C}(t)$ take values in $\mathbb{H}$, and are geodesically independent. Then the $3n$ functions*

$$j(a_1), \dots, j(a_n), \quad j'(a_1), \dots, j'(a_n), \quad j''(a_1), \dots, j''(a_n)$$

are algebraically independent over $\mathbb{C}(t)$.

With this last differential fact, we can start proving the theorem.

Lemma 5.1.6 ([FS18, Lemma 3.2])**.** *For every open $W \subseteq V$ and every finite indiscernible tuple $(a_1, \dots, a_k)$ of realisations of the type $tp(j/\mathbb{C})$, there exists an open $U \subseteq W$ such that $(a_1, \dots, a_k)$ can be realized as the restrictions to U of $(j(g_1 z), \dots, j(g_k z))$ for $g_i \in GL_2(\mathbb{C})$.*

Proof. By the Shelah reflection principle 2.1.11 we can find a canonical base $\{d_1, \dots, d_n\}$ of $\mathrm{tp}(j/\mathbb{C})$ in an initial segment of a Morley sequence. By using the Seidenberg Embedding Theorem 5.1.4, we may assume that $\{d_1, \dots, d_n\}$ is embedded in the field of meromorphic functions on some open subset $U \subseteq W$. j is a surjective analytic function from $\mathbb{H} \to \mathbb{C}$, so it follows that there are holomorphic functions $\psi_i : U \to \mathbb{H}$ such that $j(\psi_i(t)) = d_i(t)$. But $j(\psi_i(t))$ satisfies the same differential equation as $j(t)$, because the formula $\chi(y) = 0$ is in $tp(j/\mathbb{C})$":

$$\begin{aligned}
0 &= \chi(j \circ \psi_i) \\
&= S(j \circ \psi_i) + R(j \circ \psi_i)\left((j \circ \psi_i)'\right)^2 \\
&= (S(j) \circ \psi_i) \cdot (\psi_i')^2 + S(\psi_i) + R(j \circ \psi_i)(j' \circ \psi_i)^2 \cdot (\psi_i')^2 \\
&= (\chi(j) \circ \psi_i) \cdot (\psi_i')^2 + S(\psi_i) \\
&= S(\psi_i)
\end{aligned}$$

That means, if $j \circ \psi_i$ is a solution to $\chi(x) = 0$, then $S(\psi_i) = 0$. By Remark 5.1.1, all solutions of this kind are rational functions of degree one. So $j(\psi_i(t)) = j(g_i t)$ where $g_i \in \mathrm{GL}_2(\mathbb{C})$. □

Theorem 5.1.7 ([FS18, Theorem 3.7]). *We have* $\mathrm{U}(\mathrm{tp}(j/\mathbb{C})) = 1$.

Proof. We need to show that any forking extension of $\mathrm{tp}(j/\mathbb{C})$ is algebraic, which by the finite character of forking reduces to checking extensions of $\mathrm{tp}(j/\mathbb{C})$ to finitely generated differential field extensions of $\mathbb{C}$. Let $B \supseteq \mathbb{C}$ be a finitely generated differential field extension in our differentially closed field $\mathcal{U}$ such that $\mathrm{tp}(j/B)$ forks over $\mathbb{C}$. By the Shelah reflection principle 2.1.11, there is a finite Morley sequence $\{d_1, \dots, d_n\}$ in $\mathrm{tp}(j/B)$ which is dependent over $\mathbb{C}$. By Lemma 5.1.6, we realize $d_1, \dots, d_n$ as $j(g_1 t), \dots, j(g_n t)$ for some $g_i \in \mathrm{GL}_2(\mathbb{C})$. By Fact 5.1.5, as g_i and g_j are in the same coset of $\mathrm{GL}_2(\mathbb{Q})$, $j(g_i t)$ and $j(g_j t)$ are interalgebraic over $\mathbb{Q}$. As $j(g_1 t), \dots, j(g_n t)$ is a Morley sequence in $tp(j/B)$, the type $tp(j/B)$ is algebraic. □

Definition 5.1.8. Let L be a differential field extension of K. A tuple $b \in K^m$ is a differential specialization of a tuple $a \in L^m$ over K if every differential polynomial over K that vanishes at a also vanishes at b.

Now we harvest the main result:

Theorem 5.1.9 ([FS18, Theorem 3.8]). *The set C defined by the differential equation $\chi(y) = 0$ is strongly minimal.*

Proof. The equation $\chi(y) = 0$ has degree one in order three, so we need to show that any differential specialization of j over $\mathbb{C}$ satisfies no lower order differential equation, as otherwise such an equation would define a subset $D \subset C$ with $C \cap D$ and $D \setminus C$

both being infinite. By the proof of Lemma 5.1.6 and Theorem 5.1.7, and the fact that any differential specialization f satisfies the equation

$$S(f) + R(f)\,(f')^2 = 0$$

one can assume that $f = j(gt)$ for some $g \in \mathrm{GL}_2(\mathbb{C})$. Now f satisfies the hypotheses of Nishioka's Theorem 5.1.2 as f is G-automorphic with $G = \mathrm{SL}_2(\mathbb{Z})^g = g^{-1}\mathrm{SL}_2(\mathbb{Z})g$. So f does not satisfy any nontrivial order two or less equation over $\mathbb{C}$, proving that C is strongly minimal. □

Remark 5.1.10. *In general, if a function f satisfies a strongly minimal differential equation of order n over K, then the differential equation is irreducible over K and, if $F \supseteq K$ is an algebraically closed differential field extension, then the transcendence degree of the field extension $F(f, f', \dots, f^{(n)})$ is*

$$\mathrm{trdeg}(F(f, f', \dots, f^{(n)})/F) = 0 \text{ or } n,$$

depending on whether or not $f \in F$. This follows directly from the characterization of forking in Definition 2.1.5, since the differential field extension can be either forking or non-forking.

For the example of the j-function, this means:

Corollary 5.1.11. *If $F \supseteq \mathbb{C}$ is an algebraically closed differential field extension, then the transcendence degree of the field extension $F(j, j', j'')$ is*

$$\mathrm{trdeg}(F(j, j', j'')/F) = 0 \text{ or } 3,$$

depending on whether or not $j \in F$.

5.2 The many ways j is not pfaffian

For the introduction of Freitags result in the next section, we also give a version of pfaffian chains over $\mathbb{C}$:

Definition 5.2.1. [Fre21, Definition 2.2] Let $f_1, \ldots, f_n$ be complex analytic functions on some domain $U \subseteq \mathbb{C}^m$. We call $(f_1, \ldots, f_n)$ a $\mathbb{C}$-pfaffian chain if there are polynomials $p_{i,j}(u_1, \ldots, u_m, v_1 \ldots, v_i)$ with coefficients in $\mathbb{C}$ such that

$$\frac{\partial f_i}{\partial x_j}(x) = p_{i,j}\left(x, f_1(x), \ldots, f_i(x)\right) \text{ for } x \in U.$$

A chain $f_1, \ldots, f_n$ be real analytic functions on an open set $U \subseteq \mathbb{R}^m$ is called $\mathbb{R}$-pfaffian if therer are polynomials $p_{i,j}(u_1, \ldots, u_m, v_1 \ldots, v_i)$ with coefficients in $\mathbb{R}$ such that

$$\frac{\partial f_i}{\partial x_j}(x) = p_{i,j}\left(x, f_1(x), \ldots, f_i(x)\right) \text{ for } x \in U.$$

See [Kho91, Section 2.3] for more details.

Remark 5.2.2. *By the identity theorem of holomorphic functions, any $\mathbb{R}$-pfaffian chain on an open interval I extends to a $\mathbb{C}$-pfaffian chain on a neighbourhood of I with the same polynomials for the derivatives.*

5.2.1 j is not pfaffian

In this section, we give a brief overview of Freitag's result [FS18] on the j-function not being algebraic over a $\mathbb{C}$-pfaffian chain, i.e. there not being a polynomial witnessing the dependence of j and a pfaffian chain.

Theorem 5.2.3 ([Fre21, Theorem 3.1]). *Let $U \subseteq V$ be open in the fundamental domain of j. Then the j-function is not algebraic over any $\mathbb{C}$-pfaffian chain $(f_1, \ldots, f_n)$ on U.*

Proof. Let $(f_1, \ldots, f_n)$ be a pfaffian chain on U of minimal length such that j is algebraic over the field extension $F \coloneqq \mathbb{C}(f_1, \ldots, f_{n-1})$. If j is algebraic over F, we may shorten the chain, contradicting the minimality of n. Then j is transcendental and interalgebraic with f_n over F. By Pfaffianness, the field extension F is a differential field and $F(f_n)$ is a differential field extension of F with transcendence degree 1. Now as j is interalgebraic with f_n over F we have that $\operatorname{trdeg}(F(f_n, j, j', j'')/F(f_n)) \leq 2$. Then by Corollary 5.1.11, we have $F(j, j'j'') \subseteq F(f_n)$. Thus $\operatorname{trdeg}(F(j, j', j'')/F) = 1$. This violates Corollary 5.1.11, leading to a contradiction. □

By [Apo90, Section 2.7] the j-function takes real values on the imaginary axis. This allows us to be working with $\mathfrak{j} = j(i \cdot x) : (1, \infty) \to \mathbb{R}$, taking a real interval out of the complex plane. As j is complex analytic, our $\mathfrak{j}$ is real analytic as well.

Corollary 5.2.4. *For every open interval $I \subseteq (1, \infty)$, the restriction of $\mathfrak{j}$ to I is not algebraic over any $\mathbb{R}$-pfaffian chain on I.*

Proof. By Remark 5.2.2, this situation would imply that j is algebraic over $\mathbb{C}$-pfaffian chain on a complex neighbourhood of I, which cannot happen by the previous theorem. □

Remark. *Furthermore, in [Fre21, Theorem 3.3], Freitag proved with a similar argument that both the real and imaginary parts of the j-function as real functions in* *two variables are not algebraic over any paffian chain on an open domain at the same time.*

The goal of this chapter is to extend these results to the o-minimal settings of $\overline{\mathbb{R}}$ and $\mathbb{R}_{\exp}$ as in the preprint [Kes23].

For this, we expand the definition of pfaffian to a general o-minimal expansion $\mathcal{R}$ of $\overline{\mathbb{R}}$ following [MS02]. For an o-minimal field $\mathcal{R}$ and some element a, we denote by $\mathcal{R}\langle a\rangle$ the o-minimal structure generated by taking the definable closure of $\mathcal{R}$ and a.

Definition 5.2.5. Let $U \subseteq \mathbb{R}^m$ be open. An $\mathcal{R}$-pfaffian chain is a list of analytic functions $f_1, \ldots, f_n : U \subseteq \mathbb{R}^m \to \mathbb{R}$ with open neighbourhoods W_i of $\operatorname{graph}(f_1, \ldots, f_i)$ in $U \times R^{im}$ and $\mathcal{R}$-definable functions $p_{i,j} : W_i \to \mathbb{R}^m$ such that for all $x \in \mathbb{R}^m$ we have:

$$\frac{\partial f_i}{\partial x_j}(x) = p_{i,j}\left(x, f_1(x), \ldots, f_i(x)\right)$$

Fact 5.2.6 ([Spe99])**.** *The expansion of $\mathcal{R}$ by any $\mathcal{R}$-pfaffian chain is o-minimal.*

Definition 5.2.7. A function f on U is $\mathcal{R}$-definable over functions $f_1, \ldots, f_n$ on U if there is a $\mathcal{R}$-definable function $G(x, y_1, \ldots, y_n)$ such that $f(x) = G(x, f_1(x), \ldots, f_n(x))$.

A $\mathcal{R}$-pfaffian chain $f_1, \ldots, f_n$ on an interval I is called $\mathcal{R}$-minimal it has the property that the o-minimal rank:

$$\operatorname{rk}(\mathcal{R}\langle f_1, \ldots, f_n\rangle/\mathcal{R}) = n.$$

Lemma 5.2.8 ([Kes23, Lemma 1.7])**.** *Every $\mathcal{R}$-pfaffian chain $f_1, \ldots, f_n$ on I contains a $\mathcal{R}$-minimal $\mathcal{R}$-pfaffian chain, over which each f_i is $\mathcal{R}$-definable.*

Proof. Let a $\mathcal{R}$-pfaffian chain $f_1, \ldots, f_n$ on I with corresponding functions $p_{i,j}$ for the derivatives be given. Let $k \le n$ be maximal such that (f_j) for $j \le k$ is $\mathcal{R}$-minimal. Then f_{k+1} is $\mathcal{R}$-definable over $f_1, \ldots, f_k$. There is a $\mathcal{R}$-definable function $Q(x, y_1, \ldots, y_k)$ such that $Q(x, f_1, \ldots, f_k) = f_{k+1}$. Now we can remove f_{k+1} from the chain by substituting all instances of f_{k+1} in $p_{i,j}$ for $i \ge k+1$ with $Q(x, f_1, \ldots, f_k)$. Then $f_1, \ldots, f_k, f_{k+2}, f_n$ is a $\mathcal{R}$-pfaffian chain of length $n-1$, defining the same functions over $\mathcal{R}$. □

5.2.2 j is not $\overline{\mathbb{R}}$-pfaffian

In this subsection, we fix the structure $\mathcal{R}$ to be $\overline{\mathbb{R}}$. The following fact, specific to $\overline{\mathbb{R}}$, is needed.

Fact 5.2.9 ([BCR13, Proposition 8.1.8]). *On an open semialgebraic subset $U \subset \mathbb{R}^n$, for a function $f : U \to \mathbb{R}$ the following is equivalent*

- *f is analytic and semi-algebraic on U;*
- *f is analytic and algebraic (i.e. satisfying a polynomial equation) on U.*

Theorem 5.2.10 ([Kes23, Theorem A]). *Let $I \subseteq (1, \infty)$ be an open interval. Then the restriction of $\mathfrak{j}$ to I is not $\overline{\mathbb{R}}$-definable over any $\overline{\mathbb{R}}$-pfaffian chain on I.*

Proof. Let $f_1, \ldots, f_{n+1}$ be a $\overline{\mathbb{R}}$-pfaffian chain on an interval $I \subseteq (1, \infty)$ such that $\mathfrak{j}$ on I is $\overline{\mathbb{R}}$-definable as a function over $f_1, \ldots, f_{n+1}$. We may assume that n is minimal with respect to this property. As $\mathfrak{j}$ is not definable in $\overline{\mathbb{R}}$, we have $n \ge 0$. Then by Lemma 5.2.8 $f_1, \ldots, f_{n+1}$ is $\overline{\mathbb{R}}$-minimal. So there is a definable functions G such that $\mathfrak{j}$ is given as $\mathfrak{j} = G(x, f_1, \ldots, f_{n+1})$ and by o-minimaliy there is a definable H such that $f_{n+1} = H(x, f_1, \ldots, f_n, \mathfrak{j})$.

By the pfaffian condition and chain rule, the functions $\mathfrak{j}'$ and $\mathfrak{j}''$ are $\overline{\mathbb{R}}$-definable over $f_1, \ldots, f_{n+1}$ on I. After substituting f_{n+1} with $H(x, f_1, \ldots, f_n, \mathfrak{j})$, this yields a definable function F that satisfies

$$F\left(f_1(t), \ldots, f_n(t), \mathfrak{j}(t)\right) = \left(\mathfrak{j}'(t)\right), \mathfrak{j}''(t)\right) \text{ for } t \in I.$$

By analytic cell decomposition 3.1.4 and Fact 5.2.6, after shrinking I, we may assume F is analytic on an open cell C containing graph$(f_1, \ldots, f_n, \mathfrak{j})$. Then by Fact 5.2.9, we have that F satisfies

$$p\left(F\left(x_1, \ldots, x_{n+1}\right), x_1, \ldots, x_{n+1}\right) = 0$$

for a polynomial p on C. That means that F is an algebraic function on C, so $\mathfrak{j}'$ and $\mathfrak{j}''$ are algebraic over the field $\mathbb{R}(f_1, \ldots, f_n, \mathfrak{j})$ on I. As the o-minimal rank in $\overline{\mathbb{R}}$ coincides with the transcendence degree, we get from the pfaffian condition that as differential fields of functions on I

$$\operatorname{trdeg}\left(\mathbb{R}\left(f_1 \ldots, f_n, \mathfrak{j}, \mathfrak{j}', \mathfrak{j}''\right) / \mathbb{R}\left(f_1 \ldots, f_n\right)\right) = 1.$$

Since all of the f_is are real analytic, there are holomorphic extensions $\tilde{f}_i$ on a complex domain U. By the identity theorem for holomorphic functions, it follows that

$$\operatorname{trdeg}\left(\mathbb{C}\left(\tilde{f}_1 \ldots, \tilde{f}_n, j, j', j''\right) / \mathbb{C}\left(\tilde{f}_1 \ldots, \tilde{f}_n\right)\right) = 1,$$

contradicting Corollary 5.1.11. □

This yields the corollaries:

Corollary 5.2.11 ([Kes23, Corollary 2.3])**.** $\mathfrak{j}$ *is not* $\overline{\mathbb{R}}$*-pfaffian.* □

The proof for the j-function above generalizes to the following situation:

Corollary 5.2.12 ([Kes23, Corollary 2.5])**.** *Let* $g : I \to \mathbb{R}$ *be analytic on an interval* $I \subseteq \mathbb{R}$ *and have an analytic continuation that satisfies a strongly minimal algebraic differential equation over* $\mathbb{C}$ *of order* m*. Then for all* $k \leq m - 2$*, any real restriction of the* k*-th derivative of* g *is not* $\overline{\mathbb{R}}$*-definable over any* $\overline{\mathbb{R}}$*-pfaffian chain.*

Proof. Assume there is a minimal $\overline{\mathbb{R}}$-pfaffian chain $f_1, \dots, f_n$ of length n such that a restriction $\mathfrak{g}^{(k)} : I \to \mathbb{R}$ of $g^{(k)}$ with $k \leq m - 2$ is $\overline{\mathbb{R}}$-definable over $f_1, \dots, f_n$. Then we can extend the pfaffian chain to $f_1, \dots, f_n, \mathfrak{g}^{(k)}, \mathfrak{g}^{(k-1)}, \dots, \mathfrak{g}', \mathfrak{g}$. Following the proof of the Theorem verbatim, as $n + k < n + m - 1$, we can conclude with Remark 5.1.11. □

5.2.3 j is not pfaffian over $\mathbb{R}_{\exp}$

In this section, we fix $\mathcal{R}$ to be $\mathbb{R}_{\exp}$. Regardless, $\langle a \rangle$ will still denote the $\overline{\mathbb{R}}$-definable closure.

We start with the following technical lemma:

Lemma 5.2.13 ([Kes23, Lemma 3.3])**.** *Let* $f_1, \dots, f_n$ *be an* $\mathbb{R}_{exp}$ *-pfaffian chain on an interval* I*. Then there is a* $\overline{\mathbb{R}}$*-pfaffian chain* $k = (k_1, \dots, k_l)$ *on some domain* U *ranging over the variables* $x, w_1, \dots, w_n, y_1, \dots, y_n$ *and tuples* $z_1, \dots, z_n$ *such that the projection on the first* $n + 1$ *coordinates contains* $\operatorname{graph}(x, f_1, \dots, f_n)$ *and*

$$\overline{\mathbb{R}} \langle x, f_1, \dots f_n, k_i \circ (f_1, \dots, f_n) : i \leq l \rangle ,$$

is a differential field extension of $\mathbb{R}$*, where* $k_i \circ (f_1, \dots, f_n)$ *denotes the substitution of*

f_i into the w_i variable, and the transcendence degree of the extension over $\mathbb{R}$ matches the number of generators.

Proof. By induction on n, assume we have constructed an extension for the first $n-1$ elements of the chain and let $\mathcal{R}'$ be the resulting differential field with corresponding $\overline{\mathbb{R}}$-pfaffian chain k'.

Then let $f_n'(x) = h(x, f_1(x), \ldots, f_n(x))$ with h being $\mathbb{R}_{\exp}$-definable. The by model completeness, we have for graph(h) that

$$\mathbb{R}_{\exp} \models (x, w, y_n) \in \text{graph}(h) \Leftrightarrow \exists z_n F(x, w, y_n, z_n) = 0 \tag{5.2.1}$$

for a quantifier free $\mathbb{R}_{\exp}$-definable F. By [vdDM94, Proposition 5.6] F is given as

$$P\left(x, w, y_n, z_n, g_1(x, w, y_n, z_n), \ldots, g_m(x, w, y_n, z_n)\right) \tag{5.2.2}$$

where P is a polynomial, z_n a tuple of variables and $g := (g_1, \ldots, g_m)$ is a $\mathbb{R}$-pfaffian chain such that each g_i are $\mathbb{R}_{\exp}$-definable.

In particular, it is $\overline{\mathbb{R}}$-pfaffian. Thus, among all functions F, polynomials P, and $\overline{\mathbb{R}}$-pfaffian chains g of $\mathbb{R}_{\exp}$-definable functions on open domains U such that the projection on the first $n+1$ coordinates contains graph$(x, f_1, \ldots, f_n)$ satisfying 5.2.1 and 5.2.2, choose representatives such that g does not depend on y_n, if there are any such, $|z_n|$ is minimal and (k', g) is $\overline{\mathbb{R}}$-minimal (using 5.2.8).

Then subbing the original chain $f_1, \ldots, f_n$ into the w-variables yields the $\overline{\mathbb{R}}$-extension

$$\mathcal{R} := \mathcal{R}' \left\langle f_n(x), g_1\left(x, f_1(x), \ldots, f_n, y_n, z_n\right), \ldots, g_m\left(x, f_1, \ldots, f_n, y_n, z_n\right)\right\rangle$$

over which f_n' is $\overline{\mathbb{R}}$-definable using P, making this a differential field extension.

Claim. $\operatorname{trdeg}(\mathcal{R}/\mathcal{R}')$ *is equal to the number of generators.*

Assume that $\operatorname{trdeg}(\mathcal{R}/\mathcal{R}')$ is smaller. As all k_i are $\mathbb{R}_{\exp}$-definable, there is a maximal $j \leq n$ such that $f_n, g_1 \circ f, \ldots, g_j \circ f$ is $\overline{\mathbb{R}}$-independent over $\mathcal{R}'$. If $j = n$, we are done. Assume for a contradiction that $j < n$. Then there is a $\overline{\mathbb{R}}$-definable function G such that $G(x, f_1, \ldots, f_n, k_1' \circ f, \ldots, k_l' \circ f, g_1 \circ f, \ldots, g_j \circ f) = g_{j+1} \circ f$. By $\overline{\mathbb{R}}$-cell decomposition, the equality $G(x, w_1, \ldots, w_n, k_1', \ldots, k_l', g_1, \ldots, g_j) = g_{j+1}$ holds on a $\overline{\mathbb{R}}$-definable cell C whose projection on the first $n+1$ variables $\pi(C) \supseteq \text{graph(f)}$. By our choice of g, the cell C is open and so this violates the $\overline{\mathbb{R}}$-minimality of (k', g).

By the claim, we can take $k = (k', g)$. □

As the second part of our story's prerequisites, we need the following definition in our Ax-Schanuel statement.

Definition 5.2.14. A formal parametrisation of a neighbourhood of a point $p \in \mathbb{C}$ in the sense of the Ax-Schanuel-Theorem (Fact 5.2.15) is a power series of the form $t \in p + (\mathfrak{m} \backslash \{0\})$, where $\mathfrak{m}$ is the maximal ideal in $\mathbb{C}[[z_1, \ldots, z_m]]$.

Fact 5.2.15 (Ax-Schanuel for exp and j, Case $l = 1$ of [BSCFN23, Theorem E])**.** *Let $\hat{p}_1, \ldots, \hat{p}_{k+1}$ be formal parametrisations (in variables $z_1, \ldots, z_m$) of neighbourhoods of points $p_1, \ldots, p_{k+1}$ in $\mathbb{H}$. If the transcendence degree:*

$$\begin{aligned}&\operatorname{trdeg}\left(\mathbb{C}\left(\hat{p}_1, \ldots, \hat{p}_{k+1}, \exp\left(\hat{p}_1\right), \ldots, \exp\left(\hat{p}_k\right), j\left(\hat{p}_{k+1}\right), \ldots, j''\left(\hat{p}_{k+1}\right)\right)/\mathbb{C}\right)\\ &< k + 3 + \operatorname{rank}\left(\frac{\partial \hat{p}_j}{\partial \hat{z}_i}\right)\end{aligned}$$

then there exists $n \in \mathbb{Z}^k \backslash \{0\}$ such that $\sum_{i=1}^{k} n_i \hat{p}_i \in \mathbb{C}$.

Remark. *A more general version of the Ax-Schanuel result holds for fuchsian uniformizing functions of hyperbolic curves instead of j in multiple variables, see [BSCFN23, Section 8]. Combined with the strong minimality result from [CDFN22], a similar result to Theorem 5.2.17 can be established for uniformizing functions admitting a suitable interpretation over* $\mathbb{R}$*.*

With our prerequisites out of the way, we can now prove the $\mathbb{R}_{\mathrm{exp}}$ case, following a similar strategy to the $\overline{\mathbb{R}}$-pfaffian case in Theorem 5.2.10.

Theorem 5.2.16 ([Kes23, Theorem B])**.** *On any interval* $I \subseteq (1, \infty)$ *the restriction of* $\mathfrak{j}$ *to* I *is not* $\mathbb{R}_{exp}$ *-definable over any* $\mathbb{R}_{exp}$ *-pfaffian chain on* I*.*

Proof. Let $f = (f_1, \dots, f_{n+1})$ be an $\mathbb{R}_{\mathrm{exp}}$-pfaffian chain such that on I, $\mathfrak{j}$ is definable over the chain f. By the lemma above, we can obtain a corresponding $\overline{\mathbb{R}}$-pfaffian chain $k_1, \dots, k_l$ such that the resulting differential field $\mathcal{R}$ has transcendence degree $M+1$ equal to the number of generators, and f_{n+1} is $\mathbb{R}_{\mathrm{exp}}$-interdefinable with $\mathfrak{j}$ over the other M generators. This is witnessed by $\mathbb{R}_{\mathrm{exp}}$-definable functions G and H such that $\mathfrak{j} = G(t, f_1, \dots, f_{n+1}, k_1 \circ f, \dots, k_l \circ f)$ and $f_{n+1} = H(t, f_1, \dots, f_n, k_1 \circ f, \dots, k_l \circ f, \mathfrak{j})$. By analytic cell decomposition 3.1.4 and Fact 5.2.6 H and G are analytic after potentially shrinking I. By the pfaffianness and the chain rule, we have

$$h(t, f_1(t), \dots, f_n(t), k_1 \circ f(t), \dots, k_l \circ f(t), \mathfrak{j}(t)) = (\mathfrak{j}'(t)), \mathfrak{j}''(t)) \text{ for } t \in I$$

for analytic h on a cell C depending on the pfaffian chain and G and H. Then, arguing as in the proof of Lemma 5.2.13 we have that

$$\mathbb{R}_{\mathrm{exp}} \models (t, w, x, y) \in \mathrm{graph}(h) \Leftrightarrow \exists z F(t, w, x, y, z) = 0; \quad \text{with } |z| = N$$

for $F : \mathbb{R}^{1+M+3+N} \to \mathbb{R}$ a composition of a polynomial with $g_1, \dots, g_m$ a $\mathbb{R}$-pfaffian chain of $\mathcal{L}_{\exp}$ -terms. Unraveling all $\mathcal{L}_{\exp}$ -terms from the new $g_1, \dots, g_m$ potentially increased N yields a a polynomial Q such that

$$F = Q(t, w, x, y, z, \exp t, \exp w, \exp x, \exp y, \exp z).$$

Subbinging $f_1, \dots, f_n, k_1 \circ f, \dots, k_l \circ f$ into $w, \mathfrak{j}$ into x and $(\mathfrak{j}', \mathfrak{j}'')$ into y we get the following differential field extension over $\mathbb{R}$:

$$\mathbb{R}\langle t, f_1, \dots, f_n, k_1 \circ f, \dots, k_l \circ f, \mathfrak{j}(t), \mathfrak{j}'(t), \mathfrak{j}''(t), \exp(t), \exp\left(f_1\right), \dots,$$
$$\exp\left(f_n\right), \exp(k_1 \circ f), \dots, \exp\left(k_l \circ f\right), \exp(\mathfrak{j}(t)), \exp\left(\mathfrak{j}'(t)\right), \exp\left(\mathfrak{j}''(t)\right)\rangle$$

By adding additional variables $p_0, \dots, p_3$ for representing the power series $t, \mathfrak{j}(t), \mathfrak{j}'(t), \mathfrak{j}''(t)$ on I respectively we can bring this in a form that is compatible with the Ax-Schanuel Theorem 5.2.15. Note that these $p_0, \dots, p_3$ can not contribute to the transcendence degree of the extension. $\mathfrak{j}(t), \dots, \mathfrak{j}''(t)$ can increase the transcendence degree by at most 1 by pfaffianess and there are M many f_i and k_j, and z is an N tuple. So, we have the following bound for the transcendence degree:

$$\begin{aligned}
&\operatorname{trdeg}(\mathbb{R}(p_0,\dots,p_3,f_1,\dots,f_n,k_1\circ f,\dots,k_l\circ f,z,t,\\
&\exp(p_0),\dots,\exp(p_3),\exp(f_1),\dots,\exp(f_n),\exp(k_1\circ f),\dots,\exp(k_l\circ f),\exp(z),\\
&\mathsf{j}(t),\mathsf{j}'(t),\mathsf{j}''(t))/\mathbb{R})\\
\leq&4+(M-1)+N+4+(M-1)+N\\
=&2(M+N)+6\\
<&2(M+N)+7\\
\leq&\underbrace{M+N+3}_{\text{substitutions in in exp}}+\underbrace{3}_{\text{variables in j,j}',\text{j}''}+\underbrace{\operatorname{rank}\left(\frac{\partial t,\partial p_i,\partial f_i,\partial k_{i,r}\circ f,\partial z}{\partial s_j}\right)}_{\geq M+N,\text{ by minimality of the chain and }z\text{ being independent}},
\end{aligned}$$

where s_j are the variables that we are working with. Taking the the complex analytic continuations of $t,p_0,\dots,p_3,f_1,\dots,f_n,k_1\circ f,\dots,k_l\circ f$ in a suitable complex neighbourhood of a point $p\in I$ yields formal parametrisations in the sense of 5.2.14. Then applying Ax-Schanuel 5.2.15, we have that there is a non-zero linear integer combination of the variables plugged into exp that is constant. In particular, we have that

$$\operatorname{trdeg}\left(\mathbb{R}\left\langle f_1,\dots,f_n,k_1\circ f,\dots,k_l\circ f,z,p_0,\dots,p_3\right\rangle/\mathbb{R}\right)<M+N+3 \tag{5.2.3}$$

Subbing back in t for p_0 we can then by the Lemma 5.2.13 we have that the differential field extension

$$\operatorname{trdeg}\left(\mathbb{R}\left(t,f_1,\dots,f_n,k_1\circ f,\dots,k_l\circ f\right)/\mathbb{R}\right)=M$$

Then the extension by z remains a differential field with

$$\operatorname{trdeg}\left(\mathbb{R}\langle f_1,\ldots,f_n,k_1\circ f,\ldots,k_l\circ f,t,z\rangle/\mathbb{R}\right)=M+N$$

still witnessing no dependencies. Now, adding in $\mathfrak{j}(t),\mathfrak{j}'(t),\mathfrak{j}''(t)$ as $p_1,\ldots p_3$ by strong minimality of j, we have that either

$$\operatorname{trdeg}\left(\mathbb{R}\left\langle f_1,\ldots,f_n,k_1\circ f,\ldots,k_l\circ f,t,z,\mathfrak{j}(t),\mathfrak{j}'(t),\mathfrak{j}''(t)\right\rangle/\mathbb{R}\right)$$

is $M+N$ or $M+N+3$, with the former violating the minimality of the pfaffian chain and the latter violating 5.2.3. In either case, we have a contradiction. □

Corollary 5.2.17 ([Kes23, Corollary 3.8])**.** *On any interval $I\subseteq(1,\infty)$ the restriction of $\mathfrak{j}$ to I is not $\mathbb{R}_{exp}$-pfaffian on I.* □

The other corollaries from the semialgebraic case translate to other solutions of strongly minimal differential equations, for which the corresponding Ax-Schanuel result is known.

Chapter 6

Generic derivations and differential largeness

This chapter of the thesis covers joint work with Elliot Kaplan from [KK26]. We explore the connections between two notions of *tame* differential fields, namely ones with generic derivations on algebraically bounded structures introduced by Fornaserio and Terzo [FT24] and León Sánchez and Tressl's differentially large fields [LST24]. We show that both of these notions coincide in the setting of éz fields, introduced by Walsberg and Ye [WY23]. In particular, this is the case when working in the language of $\mathcal{L}_{ring} + \{0, 1, +, \cdot\}$. Furthermore, we show that an NTP_2 algebraically bounded structure stays NTP_2 after expanding by a generic derivation, see Theorem 6.4.2, and extend the same result to NATP as well, see Theorem 6.4.3.

For the remainder of this chapter, we are working with a language $\mathcal{L} \supseteq \mathcal{L}_{ring}$ extending the language of rings and an $\mathcal{L}$-structure K expanding a field of characteristic 0. δ will be a derivation on K.

6.1 Generic derivations

The first thing we have to define is the following:

Definition 6.1.1. K is *algebraically bounded* if for all $\mathcal{L}$-extensions $K^* \succcurlyeq K$ and all $B \subseteq K^*$ are $a \in K^*$, we have

$$a \in \mathrm{acl}_{\mathcal{L}}(K \cup B) \Leftrightarrow \mathrm{trdeg}(a|K(B)) = 0$$

This, in particular, means that the model theoretic algebraic closure on the left agrees with the usual algebraic closure on fields, giving the transcendence degree on the right. In particular, this means that the induced model theoretic notion of dimension agrees with the algebraic one given by the transcendence degree.

On algebraically bounded structures, we can define what it means for a derivation to be generic.

Definition 6.1.2. The derivation δ is *generic* if for all $\mathcal{L}(K)$-definable $X \subseteq K^{r+1}$ such that the projection on the first r coordinates has dimension r, then there is an $a \in K$ such that $(a, \delta a, \ldots, \delta^r a) \in X$.

In particular, the graph of the derivation is generic in the sense that it is Zariski dense. This is a common type of expansion, as it ensures that the interactions between the algebraic structure of the field and the derivation are minimal. As we will use in the setting of Section 6.4, it is straightforward to add a generic derivation to an algebraically bounded field K. Further examples of fields with generic derivations are differentially closed fields as defined after Definition 2.1.2 and closed ordered differential fields, see [Sin78], when working in the language $\mathcal{L}_{ring}$.

6.2 Differentially large fields

In order to discuss differentially large fields, we have to first talk about large fields introduced by Pop, see [Pop14] for a survey.

Definition 6.2.1. K is *large* if is existentially closed in $K((t))$. Equivalently, K is large if every K-curve with a smooth K-point has infinitely many K-points.

Using this, we can define what it means to be differentially large.

Definition 6.2.2. K is a *differentially large field* if

- K is a large field;
- for every differential field extension $(L, \delta) \supseteq (K, \delta)$, if K is existentially closed in L, then (K, δ) is existentially closed in (L, δ).

Equivalently, differentially large fields can be characterized like this:

Fact 6.2.3 ([ST24, Theorem 2.8]). *A differential field (K, δ) is differentially large if and only if*

1. *K is a large field and*
2. *for all $r > 0$, all $P \in K[X_0, \dots, X_r]$, and all nonzero $Q \in K[X_0, \dots, X_{r-1}]$, if there is $x \in K^{1+r}$ with $P(x) = 0$ and $\frac{\partial P}{\partial X_r}(x) \neq 0$, then there is $a \in K$ with*

$$P(a, \delta a, \dots, \delta^r a) = 0 \neq Q\left(a, \delta a, \dots, \delta^{r-1} a\right)$$

Note that this characterization resembles the axiomatization of a differentially closed field 2.1.2 relative to the varieties that have solutions over K. Consequently, if

K is algebraically closed, and (K,δ) differentially large, then (K,δ) is differentially closed.

With these two notions of tame differential fields given, how do they relate to each other?

A differentially closed field does not necessarily need to admit a generic derivation, as the following example shows:

Example 6.2.4. Let (K,δ) be a differentially closed field and let $C = ker(\delta)$ be the field of constants. Let $t \in K$ be transcendental over C and take $C(t)$ with the t-adic valuation ring $\mathcal{O}$. Then $(K,\mathcal{O})$ is an algebraically closed valued field with non-trivial valuation, and algebraically bounded as an $\mathcal{L}$-structure by the quantifier elimination result by Robinson [Rob77]. The maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ has (both algebraic and model theoretic) dimension 1, but there is no $a \in \mathfrak{m}$ with $\delta a = 0$ since $C^\times \subseteq \mathcal{O}^\times = \mathcal{O} \setminus \mathfrak{m}$. Hence δ cannot be generic.

As the definition of differentially large does not consider language extensions, this motivates the question we asked for our paper:

Question 6.2.5. *Assume that $\mathcal{L} = \mathcal{L}_{ring}$. If (K,δ) is differentially large, is δ generic?*

Remark 6.2.6. *In the language $\mathcal{L}_{ring}$, and assuming K is a large field with a model complete theory T, differential largeness implies that δ is generic. By [Tre05, Theorem 7.2], the model companion of the generic derivation theory T_δ is the theory of differentially large field expansion of K, thus they have to coincide.*

We were able to answer this for the larger class of éz-fields in Theorem 6.3.9.

6.3 Éz-fields

Éz-fields were introduced in [WY23] and are named partially as a pun and partially due to their connection to the étale-open topology and the Zariski topology. Let V and W be K-varieties and let $V(K)$ be the notation for the set of K-points of the variety. The following definition relies on étale morphisms, which were introduced by Grothendieck as an algebro geometric analogue of a local homeomorphism. For a smooth variety V, the map $f : W \to V$ is *étale* if and only if W is smooth and f induces an isomorphism of the tangent spaces $T_pW \to T_{f(p)}V$ at every $p \in W$. For the more technical definition, when V is not smooth, we refer to [JTWY24, Section 2.1].

Definition 6.3.1. The *étale open topology* or $\mathcal{E}_K$-topology on $V(K)$ is given by the basis consisting of images of $f(W(K)) \to V(K)$, where $f : W \to V$ are étale-morphisms of K-varieties. Equipping each $V(K)$ with a topology this way, we have a system of topologies,which lets every morphism $f : W \to V$ between K-varieties induce continuous maps $W(K) \to V(K)$. Furthermore, these maps are open (or closed) whenever f is an open (or closed) immersion.

We can rephrase being a large field in terms of the $\mathcal{E}_K$-topology.

Fact 6.3.2 ([JTWY24, Theorem C])**.** *K is large if and only if every infinite $V(K)$ has non-discrete $\mathcal{E}_K$-topology.*

Definition 6.3.3. An $\mathcal{L}_{ring}$-definable set X is called *éz* if X is a finite union of étale open subsets of Zariski closed subsets of $V(K)$.

Note that if K is not large, then every singleton is étale open, thus every subset of K is éz, giving us no useful restriction. For large fields, however, the class of éz-sets is quite well behaved.

Fact 6.3.4 ([WY23, Theorems A and B(2)]). *Suppose K is large (and perfect). Then the class of éz sets is closed under morphisms of K-varieties. In particular, this means all existentially definable $\mathcal{L}_{ring}(K)$-definable sets are éz. Furthermore, for a smooth irreducible K-variety V and $X \subseteq V(K)$ a non-empty éz set, we have that* $\dim X = \dim V$ *if and only if X has non-empty $\mathcal{E}_K$-interior in $V(K)$.*

The addition of perfect is just here for completeness sake, as we are always working in characteristic 0. If K is large, imperfect and of characteristic p, then the set of pth roots is not éz, see [WY23, Section 5] for details. Furthermore, the étale-open topology is oftentimes already familiar. If K is separably closed, then the Zariski topology and the étale open topology agree [JTWY24, Proposition 6.1]. Similarly, if K is real closed, the order topology induces the étale open topology [JTWY24, Corollary 6.12]. Lastly, if K is a non-separably closed henselian valued field, e.g. $\mathbb{Q}_p$, then the étale open topology is induced by the valued field topology [JTWY24, Corollary 6.13].

For a variety V, we let τV denote the prolongation of V, and we let π_V denote the projection map $\tau V \to V$; see [Moo22]. The prolongation is an analog of the tangent bundle TV that takes the derivatives of defining parameters into account: if $x \in V(K)$, then $\delta x \in (\tau_x V)(K)$, and when V is defined over the constant field $\ker(\delta)$, the prolongation and tangent bundle coincide. For $a = (a_1, \dots, a_n) \in K^n$, we let $\delta a := (\delta a_1, \dots, \delta a_n)$, and for $r \in \mathbb{N}$, we let $\nabla^r(a) := (a, \delta a, \dots, \delta^r a) \in K^{(1+r)n}$.

Proposition 6.3.5 ([KK26, Proposition 2.3]). *Suppose that K is a large field and let δ be a derivation on K. The following are equivalent:*

1. *For every smooth irreducible K-variety V and every éz set $X \subseteq (\tau V)(K)$ (the prolongation of K), if $\pi_V(X) \subseteq V(K)$ has $\mathcal{E}_K$-interior, then there is $a \in V(K)$*

with $(a, \delta a) \in X$.

2. *For every éz set* $X \subseteq K^{2r}$*, if* $\pi(X) \subseteq K^r$ *has* $\mathcal{E}_K$*-interior, then there is* $a \in K^r$ *with* $(a, \delta a) \in X$.

3. *For every éz set* $X \subseteq K^{r+1}$*, if* $\pi(X) \subseteq K^r$ *has* $\mathcal{E}_K$*-interior, then there is* $y \in K$ *with* $\nabla^r(y) \in X$.

4. (K, δ) *is differentially large.*

Proof. For $1 \Rightarrow 2$, just take $V = \mathbb{A}^r$. Suppose 2 holds and let $X \subseteq K^{1+r}$ be as in 3. Consider the morphism $f : \mathbb{A}^{1+r} \to \mathbb{A}^{2r}$ given by

$$(x_0, \dots, x_r) \mapsto (x_0, \dots, x_{r-1}, x_1, \dots, x_r)\,.$$

Then $f(X) \subseteq K^{2r}$ is éz by Fact 6.3.4 and $\pi(f(X)) = \pi(X)$, so 2 gives $a = (a_0, \dots, a_{r-1}) \in K^r$ with $(a, \delta a) \in f(X)$. For $y := a_0$, we have $\nabla^r(y) \in X$.

To see that (3) $\Rightarrow$ (4), we use Fact 6.2.3. Let $r > 0, P \in K[X_0, \dots, X_r]$, and $Q \in K[X_0, \dots, X_{r-1}]^{\neq 0}$. Suppose there is $x \in K^{1+r}$ with $P(x) = 0$ and $\frac{\partial P}{\partial X_r}(x) \neq 0$. Then x is a smooth K-rational point of V_P, the zero-locus of P, so we may assume that V_P is smooth and irreducible. Take $X := V_P(K) \backslash V_Q(K)$, so X is éz and $\pi(X) \subseteq K^r$ has $\mathcal{E}_K$-open interior by Fact 6.3.4 part 2. Then 3 gives $y \in K$ with $\nabla^r(y) \in X$.

Finally, suppose that (K, δ) is differentially large and let V, X be as in 11. Then τV is smooth as well, so we can use [WY23, Theorem B(1)] to take smooth irreducible disjoint subvarieties $W_1, \dots, W_n$ of τV and $\mathcal{L}_{\text{ring}}(K)$-definable $\mathcal{E}_K$-open subsets $X_i \subseteq W_i(K)$ for each i with $X = X_1 \cup \dots \cup X_n$. Then $\pi_V(X_i)$ is $\mathcal{E}_K$-open in $V(K)$ for some i, so we set $W := W_i$ for this i and we replace X with X_i. Further shrinking X, we may assume that X is a basic $\mathcal{E}_K$-open subset of $W(K)$, so X is existentially $\mathcal{L}_{\text{ring}}(K)$-definable. As X is $\mathcal{E}_K$-open in $W(K)$, we can take an elementary $\mathcal{L}_{\text{ring}}$-extension $K^* \succcurlyeq K$ containing a tuple $(x,u) \in X^*$ that is K-generic in W using Fact 6.3.4 part 2. Then x is K-generic in V and $(x,u) \in \tau V(K^*)$, so we may extend δ to a derivation $\delta^* : K^* \to K^*$ with $\delta^* x = u$; see [Jac64, Chapter IV, Theorems 14 and 18]. As (K,δ) is differentially large and K is $\mathcal{L}_{\text{ring}}$-existentially closed in K^*, the differential field (K,δ) is existentially closed in (K^*,δ^*). As X is existentially $\mathcal{L}_{\text{ring}}(K)$-definable, we find $a \in V(K)$ with $(a,\delta a) \in X$. □

With this and the fact beforehand, we can conclude:

Corollary 6.3.6 ([KK26, Corollary 2.4])**.** *If K is a large field with a generic derivation δ on K, then (K,δ) is differentially large.*

Proof. Let $X \subseteq K^{r+1}$ be an éz set with $\pi(X) \subseteq K^r$ has $\mathcal{E}_K$-interior. By Fact 6.3.4, the $\pi(X)$ has full dimension, thus by genericity of δ, there exists an $a \in K$ with $\nabla^r(a) \in x$. By our Proposition 6.3.5 3 implies 4, K is differentially large. □

For the converse direction, we have to introduce a definition given in [WY23]:

Definition 6.3.7. We say that K is an *éz-field* if K is large and every definable set is an éz-set.

As éz-fields are by definition large, we have the following:

Corollary 6.3.8 ([KK26, Corollary 2.5])**.** *Let $\mathcal{L} = \mathcal{L}_{ring}$ and K be an éz-field. If the expansion to a differential field (K,δ) is differentially large, δ is generic.*

Proof. As any definable set X is éz, and the projection $\pi(X)$ having $\mathcal{E}_K$-interior means having full dimension by Fact 6.3.4. Proposition 6.3.5 Part 3 then spells out the definition of genericity. □

Together, these two corollaries answer Question 6.2.5

Theorem 6.3.9 ([KK26, Theorem A]). *Suppose that $\mathcal{L} = \mathcal{L}_{ring}$ and that K is an éz-field. Then (K, δ) is differentially large if and only if δ is generic.*

Proof. This follows from the Corollaries 6.3.6 and 6.3.8. □

6.4 NTP$_2$ and NATP Transfer

A common strategy to study model theoretic structures is to build them up through language expansions and see what properties remain during this process. In this section, we do the same for language expansions by generic derivations, and prove that the combinatorial properties of NTP$_2$ and NATP are well behaved under this process.

For this section, we work in the setting of an algebraically bounded structure K and let $T = Th(K)$. We fix a derivation $\delta_{\mathbb{P}}$ on $\mathbb{P} = dcl_{\mathcal{L}}(\varnothing)$ (usually $\delta_{\mathbb{P}} = 0$) and define the theories $T^{\delta} = T +$ "δ extends $\delta_{\mathbb{P}}$" and $T^{\delta}_{g} = T^{\delta} +$ "δ is generic" in the language $\mathcal{L}^{\delta} = \mathcal{L} + \{\delta\}$. Some of the properties T^{δ}_{g} inherits a lot of properties from T are the following:

Fact 6.4.1. *In the setting above, the following holds:*

1. *T^{δ}_{g} exists, is consistent and complete.*

2. *For every $\mathcal{L}^{\delta}$-formula $\varphi(x)$ (x is a tuple of variables), there is an $\mathcal{L}$-formula $\psi(x_0, \ldots, x_r)$ such that*

$$T^{\delta}_{g} \models \forall x\,(\varphi(x) \leftrightarrow \psi(x, \delta x, \ldots, \delta^{r} x))$$

3. *If T is model complete, then T^δ_g is the model companion of T^δ.*

4. *If T eliminates quantifiers, then so does T^δ_g.*

5. *If T is stable, then so is T^δ_g.*

6. *If T has NIP, then so does T^δ_g.*

7. *If T is distal, then so is T^δ_g.*

8. *If T is simple, then so is T^δ_g.*

9. *If T^δ_g eliminates imaginaries, then it is rosy.*

10. *If T has $NSOP_1$, then so does T^δ_g.*

To properly attribute these results, 1 to 9 are due to Fornasiero and Terzo, with [FT24] going up to 6 and [AG25] containing 7 to 9. Part 10 is due to León Sánchez and Mohamed [LSM25], using their setting of *derivation like theories*, which can also be used for independent proofs of 5, 8 and 9. More relevant is that structurally, parts 5 to 7 are expressible in terms of sequences and can be shown quite directly using part 2. See [ACGZ22, Proposition 7.1] and [FT24, Theorem 6.2] for general criteria, as well as the proofs of [CKP23, FK21]. On the other hand, for parts 8 to 10, which are characterized in terms of independence relations, one can show that an independence relation on T^δ_g is definable from the independence relation on T preserving these properties ([FT24, LSM25]). A third class of neo-stability properties are tree properties, defined by consistency and inconsistency patterns. These have largely been unexplored except for Point's result for NTP_2 transfer for certain classes of topological fields with a generic derivation [Poi18].

Theorem 6.4.2 ([Kap23, Theorem 3.1])**.** *If T^g_δ has TP$_2$, then so does T.*

Proof. We will assume that T^δ_g has TP$_2$, witnessed by an $\mathcal{L}^\delta$-formula $\varphi(x,y)$ in tuples x and y with the partial type $\{\varphi(x,a_{i,j})\}_{j<\omega}$ being 2-inconsistent (otherwise take a suitable conjunction of instances of ϕ). By 2.2.7, we mas assume that $|x| = 1$. By fact 6.4.1 the formula φ is equivalent to formula $\psi(\nabla^r x, \nabla^s y)$, where r and s are natural numbers and ψ is a $\mathcal{L}$-formula. Replacing a_{ij} with $\nabla^s(a_{i,j})$ and extending the length of y appropriately, we may assume that $s = 1$. Thus we fix r and $\psi(x_0,\dots,x_r)$ an $\mathcal{L}$-formula such that $\psi(\nabla^r x, y)$ has TP$_2$. We may assume that this is the minimal r with this property. If $r = 0$, then ψ is already an $\mathcal{L}$-formula with TP$_2$, witnessed by the $(a_{i,j})$. Then T has TP$_2$, and we are done. Thus, we will consider the case that $r > 0$ for the remainder of the proof. First, using fact 2.2.8, fix an array $(a_{i,j})$, which is strongly $\mathcal{L}^\delta$-indiscernible.

Then for each i, j, ω we consider the sets defined by ψ and the a_i, j:

$$X_{i,j} := \{(x_0,\dots,x_r) \in K^{r+1} : K \models \psi(x_0,\dots,x_r,a_{i,j}\} \quad X^\delta_{i,j} := \{x \in K : \nabla^r x \in X_{i,j}\}$$

Rephrasing what it means to be a TP$_2$, we have for $j \neq j'$ and all i that $X^\nabla_{i,j} \cap X^\nabla_{i,j'} = \varnothing$ and $\bigcap X^\nabla_{i,f(i)} \neq 0$ for all $f : \omega \to \omega$. Now let $\pi : K^{r+1} \to K^r$ be the projection map onto the first r coordinates. Using the minimality of r, we can show:

Claim 1. *The projection $\pi(X_{i,j})$ has dimension r for all i, j.*

Proof of Claim 1. Suppose $\pi(X_{i,j})$ has dimension $< r$ on a particular pair i, j. Then we can find a polynomial $P(x_0,\dots,x_r,y)$ such that $P(x_0,\dots,x_r,a_{i,j})$ is non-zero, but vanishes on $\pi(X_{i,j})$. In particular, we then have that $P(\nabla^{r-1}x, a_{i,j}) = 0$ for all $x \in X^\nabla_{i,j}$. Thus there exists a rational function Q such that $\delta^r x = Q(\nabla^{r-1}x, \nabla^r a_{i,j})$ for all the

$x \in X_{i,j}^{\nabla}$. Then we can rewrite $\psi(\nabla^r x, a_{i,j})$ as

$$\psi(\nabla^{r-1}x, Q(\nabla^{r-1}x, \nabla^r a_{i,j}), a_{i,j})$$

in contradiction to r being minimal. □

Then we need to show:

Claim 2. *Let $f : \omega \to \omega$ be an arbitrary function and let $n > 0$, Then the set*

$$\pi(X_{0,f(0)} \cap X_{1,f(1)} \cap \dots \cap X_{n-1,f(n-1)})$$

has dimension r.

Proof of claim 2. For each $i, j < \omega$ we name constants $b_{i,j} := (a_{ni+k,j+f(k)})$ and define formulas $\theta(x, b_{i,j}) := \bigwedge_{k<n} \psi(\nabla^r x, a_{ni+k,j+f(k)})$. Then θ has TP_2 with the array $(b_{i,j})_{i,j<\omega}$. Then, by minimality of r and the previous claim, we have that θ needs to have full dimension, thus the claim holds. □

Now back to the main proof. For i and n we set $F := \pi(X_{i,0} \cap X_{i,1} \cap \dots \cap X_{i,n})$ and we let $Z_{i,n}$ be the Zariski closure of $F_{i,n}$. Then of course $Z_{0,0} \supseteq Z_{0,1} \supseteq Z_{0,2} \supseteq \dots$. As the Zariski topology is noetherian, there is a m such that $Z_{0,m} = Z_{0,n}$ for all $n \geq m$. Then this Z_m is $\mathcal{L}(a_{0,0}, \dots, a_{0,m})$-definable, and we let $Z_{i,m}$ be the respective $\mathcal{L}(a_{i,0}, \dots, a_{i,m})$-definable set using the indiscernibility of the $a_{i,j}$. In particular

$$\pi(X_{i,j_0} \cap X_{i,j_1} \cap \dots \cap X_{i,j_m}) \subseteq Z_{i,m}$$

for $i < \omega$ and all $m < j_0 < j_1 < \dots < j_m$. Now let Z be the Zariski closure of $\pi(X_{i,j_0} \cap$

$X_{i,j_1} \cap \cdots \cap X_{i,j_m}$). Then we have that $Z \subseteq Z_{i,m}$. If not, the intersection $Z \cap Z_{i,m}$ would be a proper closed subset of both Z and $Z_{i,m}$, with the latter following from the indiscernibility of $(a_{i,j})_{j<\omega}$. However, this intersection would also contain Z_{i,j_m}, contradicting the choice of m.

Furthermore $\pi(X_{i,0} \cap X_{i,1})$ has dimension $< r$ as otherwise, by the genericity of δ we could find a $a \in K$ such that $\nabla^r a \in X_{i,0} \cap X_{i,1}$, contradicting TP_2, i.e. $X^{\nabla}_{i,0} \cap X^{\nabla}_{i,1} = \varnothing$. Therefore $m > 0$ and $Z_{i,m}$ has dimension $< r$ for all i. Now for i, j define $c_{i,j} := (a_{i,0}, \ldots, a_{i,m}, a_{i,j+m+1})$ an the sets

$$Y_{i,j} := \{(x_0, \ldots, x_r) \in X_{i,j,m+1} : x \notin Z_{i,m}\}.$$

Then $Y_{i,j}$ is defined by some $\mathcal{L}$-formula $\chi(x_0, \ldots, x_r, c_{i,j})$, and we claim that this formula has $(m+1)$-TP_2, which means that $Y_{i,j_0} \cap \cdots \cap Y_{i,j_m} = \varnothing$ for all i and $j_0 < \cdots < j_m$ and that intersection $\bigcap_{i<\omega} Y_{i,f(i)} \neq \varnothing$ for all $f : \omega \to \omega$. The first part is true by construction. The second part follows from the dimension arguments above: Let $f : \omega \to \omega$ and n be given. Then the projection

$$\pi\Big(\bigcap_{i<n} Y_{i,f(i)}\Big) = \pi\Big(\bigcap_{i<n} X_{i,f(i)}\Big) \setminus \bigcup_{i<n} Z_{i,m}$$

is of dimension r by Claim 2 above. Further, this intersection is non-empty, so $\bigcap_{i<n} Y_{i,f(i)} \neq \varnothing$. Then by [KKS14, Proposition 5.7] T has TP_2, as it is witnessed by a finite conjunction of χ. □

This proof method can be generalized to similarly defined tree properties, for example, the antichain tree property, which has many similar properties to TP_2.

Theorem 6.4.3 ([KK26, Theorem 3.2]). *If T^{δ}_{g} has ATP, then so does T.*

Proof. Let's assume that T_g^δ has ATP. Then using the respective 1-variable Fact 2.2.12 and arguing as before in the TP_2 case we, we may assume that the ATP is witnessed by a formula of the form $\psi(\nabla^r x, y)$, where x is unary, $\psi(x_0, \ldots, x_r, y)$ is an $\mathcal{L}$-formula, r is minimal, and there is a strongly indiscernible set of parameters $(a_\eta)_{\eta\in 2^{<\omega}}$. As in the TP_2 case, we define

$$X_\eta := \{(x_0, \ldots, x_r) \in K^{r+1} : K \vDash \psi(X_0, \ldots, x_r, a_\eta)\}, \quad X_\eta^\nabla := \{x \in K : \nabla^r x \in X_\eta\}$$

for each $\eta \in 2^{<\omega}$, and let $\pi : K^{r+1} \to K^r$ be the projection on the first r coordinates again. With the same proof as Claim 1 above, we get that $\pi(X_\eta)$ has full dimension r for all η. For the analogue of Claim 2, we have to actually give an argument.

Claim 1. *Let $A \subseteq 2^{<\omega}$ be a finite non-empty antichain. Then the set $\pi(\bigcap_{\eta\in A} X_\eta$ has dimension r.*

Proof. Let $\nu \in A$, for $\eta = \langle i_0, \ldots, i_{m-1}\rangle$ we define

$$\nu^\eta := \nu^\frown\langle i_0\rangle^\frown\nu^\frown\langle i_1\rangle^\frown\ldots^\frown\nu^\frown\langle i_{m-1}\rangle, \quad A_\eta := (\nu^\eta)^\frown A.$$

So $A_\varnothing = A$ and for $B \subseteq 2^{<\omega}$ we have that $\bigcup_{\eta\in B} A_\eta$ is an antichain if and only if B is an antichain as well. Then for tuples $b_\eta := (a_\mu)_{\mu\in A_\eta}$ the formula

$$\theta(x, b_\eta) := \bigwedge_{\mu\in A_\mu} \psi(\nabla^r x, a_\mu)$$

witnesses ATP on the $(b_\eta)_{\eta\in 2^{<\omega}}$. Then by the minimality of r as in the proof of Claim 2 in the TP_2 case, the statement follows. ☐

Setting $n\langle 0\rangle := \langle 0, \ldots, 0\rangle \in 2^n$ for $n \in \mathbb{N}$ and $F_n := \pi(\bigcap_{i\leq n} X_{i\langle 0\rangle})$. Then let Z_n be the Zariski closure of F_n and take m with $Z_n = Z_m$ for all $n \leq m$. Then arguing as before, $m > 0$ and $\dim(Z_m) < r$. Then $\pi(X_{\eta_0} \cap \cdots \cap X_{\eta_m}) \subseteq Z_m$ for all $m\langle 0\rangle \triangleleft \eta_0 \triangleleft \eta_1 \triangleleft \cdots \triangleleft \eta_m$, using strong indiscernibility. Then for each $\nu \in 2^{<\omega}$, we can define $c_\nu := a_\varnothing, \ldots, a_{m\langle 0\rangle}, a_{m\langle 0\rangle}^\frown\nu$ and

$$Y_\nu := \{(x_0, \ldots, x_r) \in X_{m\langle 0\rangle^\frown\nu} : (x_0, \ldots, x_{r-1}) \notin Z_m\}.$$

Then Y_ν is a definable with an $\mathcal{L}$-formula $\theta(x_0, \ldots, x_r, c_\nu)$. Now with the same argument as in 6.4.2 and the analogues of both claims as shown above, it follows that θ has m-ATP, i.e. $\bigcap_{\nu\in A} Y_\nu \neq \varnothing$ for any antichain A and $\bigcap_{\nu\in C} Y_\nu = \varnothing$ for any chain C of length m, then by Fact 2.2.11, T has ATP. $\square$

Chapter 7

Conclusion

In this thesis, we present a variety of results. We'll briefly highlight some of them here and discuss their impact and the new open questions.

From Section 4, we see that the Miller dichotomy continues on, beyond T-convex structures into those with a monomial group. The corollary 4.2.16 gives a clear indication that the further model theoretic study of exponential T-convex valued fields is likely a dead end. Every model-theoretic property that might still hold in this setting has to also hold in Peano Arithmetic, which is the poster-child for non-tame theories. On the other hand, however, we have that the powerbounded case turned out rather fruitful, with results showing quantifier elimination (4.2.5), distality (4.2.12) and even a description of the definable sets (4.2.11). This maintains that powerbounded T-convex fields, even with a monomial group, are still tame and well-behaved objects suitable for further study and modification. In particular, a question emerging from the description of the definable sets was the following:

Question. *Structures in which all unary definable sets are a union of an open set and finitely many discrete sets are sometimes called d-minimal, though d-minimal* *structures are often additionally assumed to be definably complete. Our valued field is not definably complete regarding the order, as the valuation ring is bounded but has no supremum within the model $\mathcal{R}$. In [For11, Section 9], Fornasiero gives a more relaxed definition of a d-minimal structure which does not include definable completeness. Is $\mathcal{R} \models T_{\mathfrak{M}}$ d-minimal in this latter sense?*

Section 5 proved a negative result. The fact that the unrestricted real $\mathfrak{j}$ is not $\mathbb{R}_{\text{exp}}$ pfaffian (5.2.17) is a sign that while pfaffian expansions are good candidates for settings for better point counting bounds, a pfaffian expansion $\mathbb{R}_{\text{exp}}$ will not contain the unrestricted $\mathfrak{j}$ function.

Question. *Is it possible to adapt the proof of better bounds using the methods of [BNZ24] to an expansion of $\mathbb{R}$ containing an unrestricted $\mathfrak{j}$ as the graph of $\mathfrak{j}$ is still definable as a nested Rolle leaf?*

In Section 6, we have shown that for éz-fields being algebraically bounded with a generic derivation and being differentially large coincide. It remains open, if this is the case for all large fields. Further, we established a method to prove transfer results for both NTP_2 and NATP, from an underlying algebraically bounded base field to one with a generic derivation. This method will likely generalize to further properties defined through inconsistency patterns.

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